

**Section 6.2: Summary of Standard Matrix Transformations on  $\mathbb{R}^2$**   
 Linear Algebra, Fall 2009

**Main Types of Standard Matrices**

**I.** *Rotation* by angle  $\theta$  has standard matrix  $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ .

**Example:** To rotate the triangle with vertices at  $(2, 2)$ ,  $(2, 4)$  and  $(4, 1)$  by an angle of  $\pi/2$ , take the products of  $A_{\pi/2}$  with each of these points expressed as column vectors in standard position.

$$A_{\pi/2} = \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that

$$A_{\pi/2} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \quad A_{\pi/2} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad \text{and} \quad A_{\pi/2} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$$

So the points  $(2, 2) \mapsto (-2, 2)$ ,  $(2, 4) \mapsto (-4, 2)$  and  $(4, 1) \mapsto (-1, 4)$ , the original triangle  $T$  (meaning just the boundary line segments) now becomes the triangle  $T'$  composed of the lines between the transformed points.

**II.** A *dilation/contraction* transformation  $T \left[ \begin{pmatrix} x \\ y \end{pmatrix} \right] = \begin{pmatrix} rx \\ ry \end{pmatrix}$  has standard matrix  $A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ .

**Example.** These are pretty straightforward. Remember that dilation means  $r > 1$  while contraction means  $0 < r < 1$ . Essentially, any point considered as a vector is just scaled by a factor of  $r$ . *Booooooring!*

**III.** A *reflection* can be through the origin (call it  $R_0$ ) or across some line  $y = mx + b$  (say  $R_{mx+b}$ ) in which case it can take on several different forms depending on the line.

**Subtype A** Through the origin requires  $(x, y) \mapsto (-x, -y)$ , so  $R_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .

**Subtype B** A reflection about the  $x$ - or  $y$ - axis must be of the form  $(x, y) \mapsto (x, -y)$  or  $(x, y) \mapsto (-x, y)$ , respectively. The standard matrices for these reflections are

$$R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad R_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Subtype C** Reflections about the lines  $y = \pm x$  follow from the idea that the inverse of any relation in  $x$  and  $y$  is that relation where all original  $x$ s are replaced by  $y$ s and all  $y$ s by  $x$ s. Since we must have  $(x, y) \mapsto (\pm y, \pm x)$  across  $y = \pm x$ , the standard matrices for reflection in these cases are

$$R_{y=\pm x} = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

**IV.** A *scaling matrix*, say  $T_{ab}$ , stretches/shrinks a vector  $(x, y)$  by a factor of  $a$  in the  $x$ -direction and by a factor of  $b$  in the  $y$ -direction. Hence,  $(x, y) \mapsto (ax, by)$  with standard matrix (say)  $S_{ab}$  given by

$$S_{ab} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

**Example:** To see how things happen, we map points in the square with vertices  $(1, 1)$ ,  $(2, 1)$ ,  $(1, 2)$  and  $(2, 2)$  under the scaling  $T_{23}[(x, y)] = (2x, 3y)$ , or the product of  $S_{23}$  with each of these points as vectors.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \text{and so on.}$$

Under this transformation, then,  $(1, 1) \mapsto (2, 3)$ ,  $(2, 1) \mapsto (4, 3)$ ,  $(1, 2) \mapsto (2, 6)$  and  $(2, 2) \mapsto (4, 6)$ . You may want to graph both rectangles  $R$  and  $R'$  to see the effect of a scaling on an easily recognized shape.

- V. A *projection* onto the  $x$ - or  $y$ -axis must accomplish  $(x, y) \mapsto (x, 0)$  or  $(x, y) \mapsto (0, y)$ , respectively; let's say the standard matrices for this are  $P_x$  and  $P_y$ . It is only a little more complicated to get the standard matrix for projection onto a line like  $y = x$ , call the standard matrix  $P_{y=x}$ , but we show this below. The standard matrices are

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P_{y=x} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

The last matrix can be reasoned by noting that we only need to know the image of the standard basis vectors map to the line  $y = x$ . But using geometry with right isosceles triangles we can show that both vectors  $(1, 0)^t$  and  $(0, 1)^t$  map into the vector  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^t$ . Hence the matrix  $P_{y=x}$ .

- VI. Now we have several basic transformations in hand. What about some point set in  $\mathbb{R}^2$  that undergoes several transformations — a *composition* of them— in terms of standard matrices? To get the standard matrix of a composite transformation, remember that function composition, not to mention successive standard matrix multiplications by a vector, is right to left instead of left to right.

**Example** Suppose that we are interested in knowing the standard matrix of the following transformations given in respective order: Rotate by  $3\pi/4$ , reflect across  $y = x$ , then scale in the  $x$ -direction by a factor of 3 and in  $y$  by 2. The standard matrix of this composite transformation is

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{3\pi}{4} & -\sin \frac{3\pi}{4} \\ \sin \frac{3\pi}{4} & \cos \frac{3\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix}$$

Therefore, if we take the vector  $(2\sqrt{2}, \sqrt{2})^t$  in standard position, under the sequential, composite transformations it becomes the vector

$$\begin{pmatrix} \frac{3\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix},$$

in standard position.