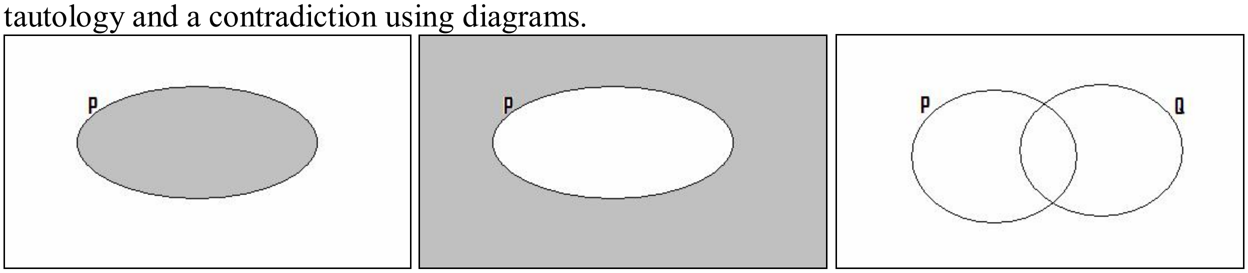
Math 3120
Homework 1: Propositions and Connectives (1.1)
Due Wednesday, August 30, **at the end of class**. **Be sure to show your work and justify your answers**.

1. Define logic. Why bother studying logic?
2. Define inductive reasoning and deductive reasoning.
3. Determine whether inductive or deductive reasoning was used.
    a. Adam has been late for school 12 times in the past month but has never been absent. Today he was not in his seat when the tardy bell rang. The teacher concluded he would arrive late.
    b. A list of numbers begins with 3, 5, 7. Alice concluded that the next number is 9.
    c. An integer is even iff it is divisible by 2. 58 is divisible by 2. Arnold concluded that 58 is even.
    d. At Inteligente College, all juniors must visit an advisor. Otherwise, they can't register for classes. Annie is a junior and she registered. Her roommate concluded Annie visited an advisor.
4. Give three different patterns that define a list of numbers beginning with 3, 5, 7.
5. Inductive reasoning cannot be used to prove something is true.
    a. Explain why.
    b. Why is inductive reasoning useful?
6. Define proposition (statement) and paradox. Note that paradox has a broader meaning than given in the text.
7. Which of the following are propositional forms (statements)? If not, explain why. #1abfh (pg 8,9)
8. Is the following sentence defining relativism a propositional form (statement)? If not, explain why. *There is no absolute truth.*
   Source: *Philosophy for Dummies*, by Tom Morris
9. Find sentences about current events in a recent news source. Please cite your sources.
    a. Give a proposition (statement).
    b. Give a sentence that is not a proposition (statement).
10. An ancient logician offended his queen and was sentenced to death. Just prior to his execution he was allowed to say one sentence. If the sentence was true, he would die by hanging. If the sentence was false, he would be beheaded. The logician's sentence was "I will be beheaded." How did he die?
    Author unknown
11. Fill in the table. Try to do this from memory.

Read	**Name**	**Symbolic representation**
P and Q		
P or Q		
not P		

12. Let p represent *I watched a movie last night* and q represent *I do not get an A.* Write in words:
    a. $\sim q$
    b. $q \wedge \sim p$
13. Give the negation in words of *Six is positive* three different ways.
14. Give a negation.
    a. $xy = 1$
    b. $-1 \leq \sin\alpha \leq 1$
    c. x and y are odd
15. What's wrong with the proposition $\sim PQ \wedge R \vee$?
16. Make truth tables for conjunction, disjunction, and negation. Try to do this from memory.
17. #3abdfik (pg 9)
18. #5ace (pg 9) Remember to write the propositional forms.
19. When is $A \wedge B \wedge C \wedge D$ true? When is it false?
20. When is $A \vee B \vee C \vee D$ true? When is it false?
21. If I prove that the negation of a statement is false, what have I proved about the statement?
22. Define tautology and contradiction.

23. #2abch (pg 9) If the statement is a tautology or a contradiction, so state.
24. If a statement includes P, Q, R, and S, how many rows will its truth table have, counting the top header row?
25. What is the negation of a tautology?
26. Create your own
    a. tautology
    b. contradiction
27. What is the notation for equivalent?
28. In many ways, equal and equivalent are similar. However, for real numbers *a* and *b*, $a = b$ means *a* and *b* are the same number, but for statements *A* and *B*, $A \Leftrightarrow B$ means *A* and *B* have the same ___________.
29. What is the order of operations for logic? (that we've learned so far)
30. Insert the "invisible parentheses" given by the order of operations.
    a. $\sim P \vee Q$.
    b. $a \wedge \sim b \Leftrightarrow c \vee d$
31. #4acegi (pg 9) What property is demonstrated in #4c? #4e? What do we learn from #4i?
32. Can negation be distributed in the following way? $\sim (p \vee q) \Leftrightarrow \sim p \vee \sim q$ Justify your answer.
33. How can knowing equivalent statements be useful in proofs?
34. If $A \Leftrightarrow \sim B$, what can I conclude about *B*?
35. If $X \Leftrightarrow Y$ and $Y \Leftrightarrow Z$, what can I conclude about *X* and *Z*? What property is this?
36. What is $\sim\sim P$ equivalent to? What is $\sim\sim\sim\sim\sim P$ equivalent to?
37. True or false? If true, what property is demonstrated?
    a. If C is equivalent to D, then D is equivalent to C.
    b. A statement is equivalent to itself.
38. Give a conjecture of equivalent symbolic statements. That is, make an educated guess of a pair of equivalent statements. You do not need to prove the conjecture.
39. Explain how *inclusive or* and *exclusive or* are different.
40. How is *or* being used in the following statements?
    a. If $ab = 0$, then $a = 0$ or $b = 0$.
    b. Dad said sternly, "Junior, clean your room or you don't get desert."
    c. A bit can take the value of 0 or 1.
       Source: http://www.bartleby.com/64/C004/010.html
    d. Paying bills on time or reducing credit card balance can improve your credit score.
       Source: http://money.cnn.com/2002/02/15/debt/q_fivethings_creditscore/
41. Please write down clearly an email address that you use regularly in case I need to contact you.
42. Bonus (2 points) Statements can represented using diagrams by shading areas to represent true. The first two diagrams below represent P and ~P, respectively, given that P is true. Given that P and Q are true, represent $P \wedge Q$ and $P \vee Q$ using diagrams. Use the third diagram as a template. Also represent a tautology and a contradiction using diagrams.



Math 3120
Homework 2: Conditionals and Biconditionals (1.2)
Due Friday, September 1, at the end of class. Be sure to show your work and justify your answers.

1. Fill in the table. Try to do this from memory.

Read	**Name**	**Symbolic representation**
if P, then Q		
P iff Q		

2. Define antecedent and consequent. What does the prefix *ante* mean?
3. Using the words *implies, sufficient, necessary, only if, unless,* and *whenever,* write *If the gas tank is empty, then the car won't run* in six different ways. Page 16 may be helpful.
4. Using the words *let, for…we have, suppose,* and *given…we have*, write *If* $x < 0$*, then* $\sqrt{x}$ *is imaginary* four different ways.
5. True or false? The antecedent of *If* $w = 0$*, then* $\frac{1}{w}$ *is undefined* is *If* $w = 0$.
6. #1acefijk (pg 17-18)
7. #3dgh (pg 18) The theorems are on pages 97, 190, and 226.
8. Write *y is irrational iff the decimal representation of y neither terminates nor repeats* in two different ways using the phrases *is equivalent to* and *is necessary and sufficient for*. Page 16 may be helpful.
9. What is the order of operations for logic? (that we've learned so far)
10. Insert the "invisible parentheses" given by the order of operations.
    a. $P \wedge Q \Rightarrow R$
    b. $P \Rightarrow Q \Leftrightarrow \sim Q \Rightarrow \sim P$
    c. $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$ (recognize this?)
11. #8bcfg (pg 19)
12. Write symbolically.
    a. a = b and b = c implies a = c. (recognize this?)
    b. Whenever $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$. (recognize this?)
13. Explain why is it incorrect to represent *Let a and b be positive. Then the product of a and b is positive* symbolically as $a \wedge b > 0 \Rightarrow ab > 0$.
14. Give an example from real life (your own life if possible).
    a. conditional statement
    b. biconditional statement
15. Make truth tables for conditional and biconditional statements. Try to do this from memory.
16. #6ac (pg 18)
17. #4abcd (pg 18)
18. #5befi (pg 18)
19. Define modus ponens (direct reasoning).
20. Suppose I know that the statement *If a function is continuous on a finite closed interval, then the function is integrable* is true and that *A function f is continuous on a finite closed interval* is true. What can I conclude?
21. A linear algebra theorem states that for an *n* x *n* matrix A, the following are equivalent:
    a. A is invertible.
    b. Ax = 0 has only the trivial solution.
    c. The columns of A are linearly independent.
    d. det A $\neq$ 0.

    Suppose a matrix B is not invertible. What can I conclude?
22. Suppose I know P implies Q and Q implies P. What can I conclude?
23. Definitions are examples of what kind of statement?
24. Let *x* be a positive integer greater than 1. Give a statement equivalent to *x is prime*.

25. Solve the following equations step by step. Connect each step with $\Leftrightarrow$ or $\Rightarrow$. The general format should be similar to the problem on the bottom of page 14, i.e., with the $\Leftrightarrow$ and $\Rightarrow$ neatly aligned vertically.
    a. $x^3 = 8x^2$
    b. $\sqrt{t^2 - 12t - 4} = 3$
26. Fill in the table. Try to do this from memory.

Name	**Symbolic**	**How to get from conditional**
conditional		NA
converse		
inverse		
contrapositive		

27. Which of the above are equivalent?
28. Which of the following statements are equivalent to *If schools aren't properly funded, then students do worse academically*?
    a. *If students do worse academically, then the schools aren't properly funded.*
    b. *If schools are properly funded, then students don't do worse academically.*
    c. *If students don't do worse academically, then the schools properly funded.*
29. Write the converse of the Pythagorean Theorem.
30. Write the contrapositive of the following:
    a. If x^2 is not divisible by 4, then x is odd.
    b. If $x + y$ is even, then either x and y are both odd or x and y or both even.
31. Given that $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$, write a statement equivalent to $B \Rightarrow \sim A$.
32. Prove Theorem 1.2ah (pg 15)
33. Prove that the converse and inverse are equivalent by using that the conditional and contrapositive are equivalent.
34. Prove $\sim (P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$ by showing it is a tautology.
35. Given that $(P \Rightarrow Q) \Leftrightarrow (\sim P \vee Q)$, write statements equivalent to the following:
    a. $\sim Y \Rightarrow X$
    b. $\sim A \vee \sim B$
36. Write four De Morgan's laws (distributive laws). Try to do this from memory.
37. Using De Morgan's laws, write statements equivalent to the following:
    a. $\sim (\sim Q \vee P)$
    b. $\sim C \wedge \sim D$
    c. $\sim C \vee D$
    d. $Q \wedge (P \vee \sim Q)$
    e. $(\sim P \vee \sim R) \wedge (\sim P \vee S)$
38. Read about Augustus De Morgan at http://web01.shu.edu/projects/reals/history/demorgan.html. Write something interesting you learned. Please be aware that not everything on the internet is factual.
39. Prove Theorem 1.2e from Theorem 1.2a (pg 15).
40. #9adf (pg 19) Do not use truth tables to prove these. Show your scratch work.
41. Bonus (2 points) In this section we use *if and only if, necessary and sufficient*, and *equivalent* for the biconditional. The text actually defines the biconditional using its truth table, while *if, only if, necessary, sufficient*, and *equivalent* were defined previously. Explain why uses of *if and only if, necessary and sufficient*, and *equivalent* are justified.

Math 3120
Homework 3: Quantifiers (1.3 Part I)
Due Wednesday, September 6, at the end of class. Be sure to show your work and justify your answers.

1. Define natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, and complex numbers. Give the symbol if applicable. Try to do this from memory.
2. List which of the numbers $17, -\frac{17}{8}, 6-i, 3\pi, \sqrt{9}, 0, -1.\overline{245}, \sqrt{-9}, -101, \frac{2}{\sqrt{6}}$, and 5.4 are
   a. natural numbers
   b. whole numbers
   c. integers
   d. rational numbers
   e. irrational numbers
   f. real numbers
   g. complex numbers
3. What's wrong with the following definitions of $\mathbb{Q}$? What numbers are missing or superfluous?
   a. The set of numbers which can be expressed in the form $\frac{a}{b}$ where a and b are numbers and $b \neq 0$.
   b. The set of numbers which can be expressed in the form $\frac{a}{b}$ where a and b are whole numbers and $b \neq 0$.
4. Give an a and b for the following rational numbers. If a number is not rational, so state.
   $\frac{2}{3}, -\frac{2}{3}, 5\frac{4}{7}, 16, 0.3, 0.\overline{3}, \frac{\sqrt{6}}{\sqrt{24}}, \frac{9\pi - 8e}{8e - 9\pi}, 0$
5. Given a rational number and its representation $\frac{a}{b}$, are a and b unique? Explain.
6. Define open sentence, truth set, universe, quantifier, and counterexample.
7. Give the truth set of the following statements. The universe is the real numbers.
   a. $x^2 + x - 1 = 0$
   b. $x^2 + x + 1 = 0$
   c. $-2x + 8 < 7 + 12x \leq 11 - 2x$
8. True or False?
   a. All snakes are reptiles.
   b. There are no reptiles that are not snakes.
   c. Some rational numbers are irrational numbers.
   d. Some rational numbers are not irrational numbers.
9. Match each statement to an equivalent statement. Try to do this without looking at your notes.

All P are Q. ________	a. All P are not Q.
No P are Q. ________	b. Not all P are Q.
Some P are Q. ________	c. There are no P that are not Q.
Some P are not Q. ________	d. There exists at least one P that is Q.

10. Fill in the table. Try to do so from memory.

Statement	Negation	Equivalent Negation
All P are Q.		
Some P are Q.		

11. True or False?
   a. A negation of *Some of the apples are red* is *Some of the apples are not red.*
   b. A negation of *All math problems are difficult* is *No math problems are difficult.*
   c. A negation of *Some X are not Y* is *Some Y are not X.*
   d. A negation of *⅔ of the students are freshmen* is *⅔ of the students are not freshmen.*

12. Write equivalent statements.
    a. At least one subset is empty.
    b. There are no subsets that are not proper.
    c. Not all solutions are valid.
    d. All Republicans are conservatives.
    e. Some functions are not continuous.
13. Write negations of the following statements.
    a. All my ex's live in Texas. Source: *All My Ex's Live in Texas*
    b. Some people say a man is made out of mud. Source: *Sixteen Tons*
    c. Nobody knows de trouble I've seen. Source: *Nobody Knows de Trouble I've Seen*
    d. There exists at least one matrix which is invertible.
    e. All solutions are not unique.
    f. Not all rectangles are squares.
    g. The fourth root of a negative number is never defined.
14. Fill in the table. Try to do so from memory.

Read	Symbol	Quantifier
there exists		
for every		
there exists a unique		NA
such that		NA
element of		NA

15. Let *B* be a set, which is a collection of objects. Write $a \in B$ in words and explain what it means.
16. Write in words.
    a. $\forall a, b, c \in \Re, \quad a(b+c) = ab + ac$. (recognize this?)
    b. $\exists m \ s.t. \ m \leq 0 \wedge m \geq 0$.
    c. A function $f : X \to Y$ is **onto** iff $\forall y \in Y, \exists x \in X \ s.t. \ f(x) = y$.
    d. $\forall action, \exists \ equal \ and \ opposite \ reaction$. (recognize this?)
    e. $\exists! x \ s.t. \ g(x) = 0$.
17. Write symbolically.
    a. For every $x, |x| \geq 0$.
    b. *a* **divides** *b* iff there exists an integer *n* such that $b = an$.
    c. There exists a unique *y* such that $2^y = 3$.
    d. For every *t*, if $t < 0$ then $t + t < 0$.
    e. A function $f : X \to Y$ is **bounded** iff there exists a real number M such that for all $x \in X$, $|f(x)| \leq M$.
    f. $f : X \to Y$ is **continuous** at $a \in X$ iff for every $\varepsilon > 0$, there exists a $\delta > o$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.
18. Write what $\exists! x \ s.t. \ Ax = b$ really means, both in words and symbolically.
19. What's wrong with the statement *For* $\forall x, x \leq x$?
20. Create your own statements using
    a. $\exists$
    b. $\forall$
    c. $\exists!$
21. Explain the difference in meaning between the following two statements:
    - For every woman, there exists a man such that the woman can love the man.
    - There exists a man such that for every woman, the woman can love the man.
22. Bonus Draw a Venn diagram with the natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, and complex numbers.

Additional resource used in creating this assignment: *Thinking Mathematically* by Robert Blitzer

Math 3120
Homework 4: Quantifiers, Direct Proofs (1.3 Part II, 1.4 Part I)
Due Friday, September 8, at the end of class. Be sure to show your work and justify your answers.

1. Fill in the table. *P(x)* represents a statement involving *x*. Its negation is represented by *~P(x)*. *P(x,y)* represents a statement involving *x* and *y*.

Statement (symbolic)	Statement in words using some or all.	Negation in words using some or all.	Negation (symbolic)
$\forall x, P(x)$.			
$\exists x\ s.t.\ P(x)$.			
$\forall x, \exists y\ s.t.\ P(x,y)$.			
$\exists x\ s.t.\ \forall y, P(x,y)$.			

2. Give the negation, both in words and in symbols, of $\exists! x\ s.t.\ P(x)$.
3. True or False? If false, give which part(s) of $\exists! x\ s.t.\ P(x)$ failed.
   a. $\exists!(x,y)\ s.t.\ (x,y)$ satisfies $\begin{cases} -4x-2y=-12 \\ y=-2x+6 \end{cases}$.
   b. $\exists!(x,y)\ s.t.\ (x,y)$ satisfies $\begin{cases} x+y=6 \\ 2x+y=6 \end{cases}$.
   c. $\exists!(x,y)\ s.t.\ (x,y)$ satisfies $\begin{cases} y=6-x \\ -2x-y=-3 \end{cases}$.
4. Give a negation.
   a. $\forall \alpha, -1 \le \sin\alpha \le 1$.
   b. $\exists t\ s.t.\ e^t = 0$.
   c. $\forall x \ne 0, \exists y\ s.t.\ xy = 1$. (recognize this?)
   d. $\exists z\ s.t.\ \forall y, zy = y$. (recognize this?)
   e. $\exists! z\ s.t.\ z^2 = 9$.
   f. [There are] two girls for every boy. Source: *Surf City*
5. To prove statement P is false, what do I prove about ~P?
6. A triangle is acute iff all of its angles are less than 90°. How would I prove a triangle is not acute?
7. A function $f: X \to Y$ is **onto** iff $\forall y \in Y, \exists x \in X\ s.t.\ f(x) = y$. How would I prove a function *f* is not onto?
8. A function $f: X \to Y$ is **bounded** iff there exists a real number M such that for all $x \in X$, $|f(x)| \le M$. How would I prove a function *f* is not bounded?
9. Do $\exists$ and $\forall$ commute? That is, is $[\forall x, \exists y\ s.t.\ P(x,y)] \Leftrightarrow [\exists y\ s.t.\ \forall x, P(x,y)]$?
10. Classify the statements as true or false. If a statement involving $\forall$ (all) is false, give a counterexample. If a statement involving $\exists$ (some) is true, give an example.

Statement	**Universe**	**T/F**	**Counterexample or example (if applicable)**
All people are taller than 5 feet.	Americans		
All people are taller than 5 feet.	National Basketball Association players (NBA)		
$\forall x, x^2 + 1 \ge 0$.	$\mathbb{C}$		
$\exists x\ s.t.\ x^2 - 4 = 0$	irrationals		
All irrational numbers are of the form $\sqrt{x}$.	$\mathbb{N}$		
No even numbers are prime.	$\mathbb{Z}$		

$\exists z \ s.t.\ z^2 + z + 41$ is prime.	$\mathbb{N}$		
$\forall z, z^2 + z + 41$ is prime.	$\mathbb{N}$		
$\forall j, \exists k \ s.t.\ k < j$. (recognize this?)	whole numbers		
$\forall p$ and $\forall q$ with $p < q, \exists r \ s.t.\ p < r < q$. (recognize this?)	$\mathbb{Q}$		
$\forall a, \exists! b \ s.t.\ a = b^2$.	$\mathbb{R}$		

Source: http://www.nba.com/news/survey_height_2004.html#bottom

11. Give the universes, if any, for which the statements are true. Choose from $\mathbb{C}$, $\mathbb{R}$, the irrational numbers, the whole numbers, and $\mathbb{N}$.
    a. $\forall x, 3x \ge x$.
    b. $\exists s \ s.t.\ 8s - 5 = 6 - 2s$.
    c. $\forall a, \exists b \ s.t.\ a + b = 0$. (recognize this?)
    d. $\exists q \ s.t.\ \forall p, q + p = p$. (recognize this?)
12. True or False? *All of the live elephants in the Math 3120 classroom are pink*. Explain.
13. Why bother to study proofs? Aren't some things obviously true? Didn't a bunch of ancient mathematicians already prove all this stuff anyway?
14. If we show that when P is true, Q is never false, what can we conclude about $P \Rightarrow Q$? Why?
15. Briefly explain how to prove something using a direct proof.
16. Define odd and even.
17. Using the definitions of odd and even, show
    a. 17 is odd
    b. $2xy + 4x - 3$ is odd (x and y are integers)
    c. -10 is even
    d. $4c^2+2$ is even (c is an integer)
18. #5ae and #7c (pg 37) Please show your scratch work off to the side and clearly label it as such. You may use the definitions of odd and even. You may use that the sum, difference, and product of integers is an integer. You may also use basic arithmetic and the properties of real numbers, such as the distributive and associative properties.
19. Give an application for each of the theorems proved in #5ae and #7c (pg 37). For example, we have for #5d (pg 37)
    Theorem: Let x and y be integers. If x is even and y is odd, then $x + y$ is odd.
    Application: 6 is even and 43 is odd. By the theorem, 6 + 43 = 49 is odd.
20. Create and prove your own theorem relating to odd and/or even. It doesn't need to be fancy.
21. What's wrong with the following proof?
    Theorem: Let m be an integer. If m is odd, then m^2 is also odd.
    Proof: 7 is an odd integer. $7^2 = 49$ is odd. Therefore, if an integer m is odd, then m^2 is also odd. ■
22. List what is commonly used in proofs for the following:
    a. the statement of what you will prove, such as theorem
    b. connective words, such as then
    c. symbols to signify the end of the proof, such as ■
23. Bonus True or False? Justify your answer. These problems address how to do direct proofs involving $\wedge$ and $\vee$. For example, the first one asks if in order to prove P implies Q or R, we can prove P implies Q or P implies R.
    a. $[P \Rightarrow (Q \vee R)] \Leftrightarrow [(P \Rightarrow Q) \vee (P \Rightarrow R)]$
    b. $[P \Rightarrow (Q \wedge R)] \Leftrightarrow [(P \Rightarrow Q) \wedge (P \Rightarrow R)]$
    c. $[(P \vee Q) \Rightarrow R] \Leftrightarrow [(P \Rightarrow R) \vee (Q \Rightarrow R)]$
    d. $[(P \wedge Q) \Rightarrow R] \Leftrightarrow [(P \Rightarrow R) \wedge (Q \Rightarrow R)]$

Math 3120
Homework 5: Direct Proofs, Exhaustion, Working Backwards, Previous Results (1.4 Part II, 1.6 Part I)
Due Monday, September 11, at the end of class. Be sure to show your work and justify your answers. Please show your proof scratch work off to the side and clearly label it as such.

1. For integers a and b, define a divides b.
2. True or False?
   a. $18|144$
   b. $144|18$
   c. $18/144 = 18\,|\,144$
   d. $144/18 = 18\,|\,144$
3. Given that a divides b, fill in the blanks with a or b.
   a. __ | __
   b. __ is a divisor of __
   c. __ is divisible by __
   d. __ is a factor of __
   e. __ divided by __ has remainder 0.
   f. __ = __ n for some integer n.
4. Using the definition of divides, show
   a. 14 divides -84
   b. p divides $z = p^2 - akp$
5. Rewrite the following theorems with the hidden "for every's." Then give the first sentence of the proof.
   a. Let x and y be integers. Then if x and y are even, then $x + y$ is even.
   b. Suppose a is an integer. Then $2a - 1$ is odd.
   c. The conjugate of the conjugate of $a + bi$ is $a + bi$.

In the proofs below you may use

- *the definitions of odd, even, divides, and absolute value*
- *the sum, difference, and product of integers is an integer*
- *basic arithmetic and the properties of real numbers, such as the distributive and associative properties*
- *basic algebra manipulations, such as adding the same number to both sides of an equation*
- *the quadratic formula*
- *the product of two negative numbers is negative and similar results*
- *the square of a real number is never negative*
- *rules of inequalities, such as dividing by a negative number causes the inequality to be reversed*
- *the Pythagorean Theorem and its converse.*

6. Suppose a, b, and c are integers. Prove that if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$. Also give an application.
7. #7gk (pg 38) Also give applications of these.
8. #2a (pg 53) Also give an application.
9. Using #2a (pg 53), show that if $a \mid b$ and $a \mid c$, then $a \mid (2b + 3c)$, $a \mid (b - c^2)$, and $a \mid 2bc$.
10. Define $|\,x\,|$ where x is a real number.
11. Give | a | if
    a. $a < 0$
    b. $a > 0$
    c. $a = 0$
12. When is $b \le -b$?
13. If $a + b < 0$, what is $|a + b|$?
14. If $a < 0$ and $b > 0$, what sign is $a + b$?
15. Explain what is meant by a proof by exhaustion.
16. When can you use the phrase *The case … is similar* in a proof?
17. #7d and #6ad (pg 37) (recognize #6d?) Also give applications of these.
18. Explain what is meant by "working backwards" when proving something.

19. #9ace (pg 38) Also give applications of these.
20. Let $a = 2$, $f(x) = 3x + 1$, and $\delta = \frac{\varepsilon}{3}$. Prove that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.
21. Let $N = \frac{1}{\varepsilon^2}$, $f(x) = \frac{1}{\sqrt{x}}$, and $L = 0$. Prove that $|f(x) - L| < \varepsilon$ whenever $x > N$.
22. #11 (pg 38,39)
23. How can previously proven results be used in proofs?
24. #8 (pg 38) Prove this using first exhaustion and then again using previous results. Give an application.
25. #1g and #2b (pg 53) Also give applications of these.
26. What's wrong with the following proof?
    Theorem: If $a + 1$ is even, then a is odd.
    Proof: $a + 1 = 2k$
    $a = 2k - 1 = 2(k - 1) + 1$■
27. Bonus Show the alternate definitions of odd and even below are equivalent to those given in class, i.e., show how to get from one definition to the other and vice versa.
    a. A number is even iff 2 divides the number.
    b. A number is odd iff it can be expressed as $2m - 1$ where m is an integer.

Math 3120
Homework 6: Proof by Contraposition, Iff Proofs, Proof by Construction (1.5 Part I, 1.6 Part II)
Due Wednesday, September 13, at the end of class. Be sure to show your work and justify your answers.
Please show your proof scratch work off to the side and clearly label it as such.

1. Explain how to prove something by contraposition. Why does it work?
2. #3ce (pg 45) Be sure to write the contrapositive at the beginning of the proof. Give applications of these.
3. #4a (pg 45) Be sure to write the contrapositive at the beginning of the proof.
4. Explain the two part proof of an iff statement. Why does it work?
5. #7c (pg 45) Give an application.
6. #6e (pg 37) You may assume that $b > 0$.
7. Explain how to prove something by construction.
8. Prove $\exists x \in \Re \ s.t.\ x \geq 0$ and $x \leq 0$.
9. #1a (pg 53) Do we have uniqueness?
10. #5b (pg 54) Give an application.
11. Prove there is a real solution to $-2x^4 + 18x^3 - 16x^2 - 9x + 9 = 0$ first by construction and then by using the Intermediate Value Theorem. You do not need to factor the polynomial.
12. Prove there exists a natural number that is the sum of its positive divisors other than itself (proper divisors). Numbers such as these are called perfect numbers.

In the proofs below, you may use basic manipulations with absolute value, such as $|ab| = |a||b|$.

13. #7c (pg 54) Give an application.
14. The infamous ε-δ proofs from calculus are construction proofs. $f : X \to Y$ is **continuous** at $a \in X$ iff for every $\varepsilon > 0$, there exists a $\delta > o$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Prove the following functions are continuous at *a*.
    a. $f(x) = 2x,\ a = 1$
    b. $f(x) = x^2,\ a = 0$
15. Limits also involve construction. $\lim_{x \to c} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta > 0 \ s.t.\ |f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$.
    Prove that $\lim_{x \to 2} 5x - 1 = 9$
16. A limit at positive infinity is defined by $\lim_{x \to +\infty} f(x) = L$ iff $\forall \varepsilon > 0, \exists N > 0 \ s.t.\ |f(x) - L| < \varepsilon$ whenever $x > N$. Prove the following limits:
    a. $\lim_{x \to +\infty} \frac{1}{x} = 0$
    b. $\lim_{x \to +\infty} 2^{-x} = 0$
17. Prove that it is possible for a human being to run a mile in less than four minutes.
18. Bonus: Prove $\forall p \in Q$ and $\forall q \in Q$ with $p < q, \exists r \in Q \ s.t.\ p < r < q$. This shows what's called the density of the rational numbers. Give an application.

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Homework 7: Proof by Contradiction, Proving Uniqueness (1.5 Part II, 1.6 Part III)

Due Friday, September 15, at the end of class. Be sure to show your work and justify your answers. Please show your proof scratch work and clearly label it as such.

1. Explain how to prove something by contradiction.
2. #6ad and #9 (pg 45) Please give applications.
3. Do #9 (pg 45) by working backwards and then giving a direct proof.
4. #10 (pg 46)
5. #1c and #2d (pg 53)
6. #7e (pg 54)
7. Explain how to prove uniqueness.
8. Prove that every real number has a unique additive inverse. That is, prove $\forall a \in R, \exists! b \in R \; s.t. \; a + b = 0$.

You may use basic matrix algebra manipulations such as Ax – Ay = A(x – y) in the following proof.

9. Prove the following: Suppose that for an $n \times n$ matrix A, the equation $Ax = 0$ has only the zero solution, i.e., $x = 0$. Here 0 represents the $n \times 1$ vector with all zero entries. Then for any $n \times 1$ vector b, the solution to $Ax = b$ is unique.
10. List the methods of proof we've learned so far.
11. Which methods of proof are used in the following proofs?
    a. Theorem: Let x_1 and x_2 be solutions to $Ax = 0$. Then $x_1 + x_2$ is also a solution.
       Proof: Let x_1 and x_2 be solutions to $Ax = 0$. Then $Ax_1 = 0$ and $Ax_2 = 0$. Subtracting the equations we get $Ax_1 - Ax_2 = 0$. Then $A(x_1 - x_2) = 0$ and $x_1 + x_2$ is also a solution to $Ax = 0$.■
    b. Proposition: Let n be an integer. Then $n^2 + n + 2$ is even.
       Proof: Case 1: Suppose n is even. Then $n = 2k$ for some integer k. We have that $n^2 + n + 2 = (2k)^2 + 2k + 2 = 4k^2 + 2k + 2 = 2(2k^2 + k + 1)$, which is even.
       Case 2: Suppose n is odd. Then $n = 2k + 1$ for some integer k. We have that $n^2 + n + 2 = (2k + 1)^2 + 2k + 1 + 2 = 4k^2 + 4k + 1 + 2k + 1 + 2 = 4k^2 + 6k + 4 = 2(2k^2 + 3k + 2)$, which is even.
       Hence $n^2 + n + 2$ is even. ∴
    c. Lemma: If xy is odd, then x and y are odd.
       Proof: We shall prove the contrapositive, namely, if x or y is even, then xy is even. First we suppose x is even. Then there is an integer n such that $x = 2n$. Then $xy = 2ny$, which is even. The case where y is even is similar. We conclude that if xy is odd, then x and y are odd. ■
    d. Conjecture: There exist two irrational numbers whose sum is a rational number.
       Proof: The irrational numbers $-\sqrt{3}$ and $\sqrt{3}$ sum to 0, which is rational. *QED*
    e. Claim: $\sqrt{2}$ is irrational.
       Proof: Suppose not. Then $\sqrt{2} = \frac{p}{q}$ where p and q ares integers and q is nonzero. Without loss of generality, we also make the restriction that p and q have no common factors other than ± 1. By squaring both sides, we have that $2 = \frac{p^2}{q^2}$, i.e., $2q^2 = p^2$. Then p^2 is even. By a previous result, p is even. Then $p = 2j$ for some integer j. Then $2q^2 = (2j)^2 = 4j^2$. So $q^2 = 2j^2$. Then q is even by the same argument used above. However, this means p and q both have 2 as a factor, and we have a contradiction. We conclude that $\sqrt{2}$ is irrational.□
12. Prove the following. State which method(s) of proof you use for each.
    a. $\exists t \; s.t. \; t^3 - t^2 - 4t + 4 = 0$
    b. If ab is odd, then both a and b are odd.
    c. If $2 < x < 3$, then $x^2 - 5x + 6 < 0$.
    d. $|a - b| = |b - a|$
    e. m is odd iff m^2 is odd.

13. What is your favorite method of proof? least favorite?
14. Bonus: A matrix A has an inverse B iff $AB = I$ and $BA = I$, where I is the matrix identity. Prove that if a matrix A has an inverse, then it is unique.