Lecture 1: Propositions and Connectives (1.1) Supplement

What is the difference between the reasoning processes in the following two examples?

- The past six telephone calls Alfonzo received were from telemarketers. Alfonzo doesn't answer the next time the phone rings, as he concludes that it is a telemarketer calling.

- If Anita's alarm clock has the power cut, then the clock needs reset. The power went out for an hour. Anita concludes that she will need to reset the clock.

Which of the following are propositions? If not, explain why

- Iraq developed biological weapons.
- $7 < 0$
- $5x - 3 = x + 7$
- Gosh!
- Is it raining?
- It is raining.
- Math 3120 is excruciatingly fun.
- He will get an A in Math 3120.
- This sentence is false.

Read	**Name**	**Symbolic representation**
P and Q		
P or Q		
not P		

Let *B* represent the statement *Beans are not green* and let *C* represent the statement *Carrots are orange.* Write the following in words.

a. $B \wedge C$

b. $\sim C$

c. $\sim C \vee B$

Let *p* represent the statement *Five is even, q* represent the statement *2+4=6,* and let *r* represent the statement *3 > 0.* Write the following symbolically.

d. Five is even and 3 > 0.

e. 2+4=6 or five is not even.

f. It is not the case that 3 > 0.

What happened	**What I said**	**Did I tell the truth?**
Ardith ate both pizza and goulash.	Ardith ate pizza or goulash.	
Ardith ate pizza only.	Ardith ate pizza or goulash.	
Ardith ate goulash only.	Ardith ate pizza or goulash.	
Ardith ate neither pizza nor goulash.	Ardith ate pizza or goulash.	

What happened	**What I said**	**Did I tell the truth?**
The punishment was both cruel and unusual.	The punishment was cruel and unusual.	
The punishment was cruel but not unusual.	The punishment was cruel and unusual.	
The punishment was unusual but not cruel.	The punishment was cruel and unusual.	
The punishment was neither cruel nor unusual.	The punishment was cruel and unusual.	

How are the following different?

- Andy will marry Abby or Anya.
- Amos will have Fruit Loops or granola for breakfast.

Which or?

- Attila will read a book or watch a movie.
- If $ab < 0$, then $a < 0$ or $b < 0$.

Review of Properties (these can be generalized to more than the real numbers and addition and multiplication)
Axioms of equality

- Reflexive axiom: $a = a$.
- Symmetric axiom: $a = b$ implies $b = a$.
- Transitive axiom: $a = b$ and $b = c$ implies $a = c$

Axioms of operations:

- Commutative axioms: $a + b = b + a$; $a \cdot b = b \cdot a$.
- Associative axioms: $a + (b + c) = (a + b) + c$; $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Distributive axioms: $a \cdot (b + c) = a \cdot b + a \cdot c$; $(b \cdot c)\ n = b\ n \cdot c\ n$.
- Identity axioms: $a + 0 = a$; $a \cdot 1 = a$; $a\ 0 = 1$.
- Inverse axioms: $-a$ exists and has the property that $a - a = 0$; if a is non-zero, then $1/a$ exists and has the property that $a \cdot (1/a) = 1$.

Source: http://en.wikipedia.org/wiki/Number_system

Changes from and additions to the book:

- Often we use the word *statement* rather than *proposition*.
- The book considers sentences with a variable (such as x) or with a pronoun (such as he) to not be propositions. We assume that a particular value is assigned to the variable or that a particular person is represented by the pronoun, so that the truth value can be determined and the sentence is indeed a proposition.
- We define paradox as a sentence that is true and false simultaneously.
- We use the word negation rather than denial.
- We use the notation $\Leftrightarrow$ for equivalent.

Lecture 2: Conditionals and Biconditionals (1.2) Supplement

Read	**Name**	**Symbolic representation**
if P, then Q		

Give the antecedent and consequent in the following examples: (see pg 16 for help)

- If you get 93%, then you have an A.

- Sanctions against the Hamas government will be lifted if it denounces violence and recognizes Israel.

- Whenever winds reach 155 mph, a hurricane is classified as category 5.
  Source: http://academic.brooklyn.cuny.edu/geology/leveson/core/topics/storm_surge/hurri_def.html

- Rainy weather implies that the picnic will be canceled.

- A maximum cannot occur on an open interval unless the function has a critical point.

- Being charismatic is necessary to be a politician.

- x = 3 only if $x^2 = 9$.

- Drinking three cups of milk a day is sufficient in meeting one's calcium needs.

- Principle of Mathematical Induction (pg 97)
  If S is a subset of $\mathbb{N}$ with these two properties:
  i. $1 \in S$
  ii. for all $n \in \mathbb{N}$, if $n \in S$, then $n+1 \in S$,
  then $S = \mathbb{N}$.

- Suppose x and y are even. Then xy is divisible by 4.

- Given that x and y are odd, we have that $x + y$ is even.

Rewrite *Suppose x and y are even. Then xy is divisible by 4* using *let* and *for…we have*.

Read	**Name**	**Symbolic representation**
P iff Q		

Write using logical connectives (write symbolically).

- a = b implies b = a (recognize this?)

- x > 0 is necessary and sufficient for 2x > 0

- Suppose x = 1 or x = -1. Then |x| = 1. (#8d pg 19)

What happened	**What I said**	**Did I tell the truth?**
Bill called and the phone rang.	If Bill calls, then the phone rings.	
Bill called and the phone didn't ring.	If Bill calls, then the phone rings.	
Bill didn't call and the phone rang.	If Bill calls, then the phone rings.	
Bill didn't call and the phone didn't ring.	If Bill calls, then the phone rings.	

Let P be the statement *An animal is a snake* and Q be the statement *An animal is a reptile*.

Is $P \Rightarrow Q$ true?

Is $Q \Rightarrow P$ true?

Let P be the statement *Two lines have the same slope* and Q be the statement *Two lines are parallel*.

Is $P \Rightarrow Q$ true?

Is $Q \Rightarrow P$ true?

Suppose I know that *An animal is a herbivore iff the animal feeds chiefly on plants*.

I come across an animal that feeds chiefly on plants. What can I conclude?

I come across an animal that is not a herbivore. What can I conclude?

Source: http://www.yourdictionary.com/ahd/h/h0158500.html

Name	Symbolic	How to get from conditional
conditional		NA
converse		
inverse		
contrapositive		

Proper proof format (A well written proof is a work of art!)

Do scratchwork off to the side.

Proof:

(Using what you know, show step by step how to conclude the theorem is true. Give a justification for each step unless it is really obvious. Write in complete sentences. Use correct spelling, grammar, and notation. When appropriate, use words like thus, therefore, hence, then, we conclude, etc. Equivalent signs should be aligned vertically if more than one line is used.)

(A symbol such as ■, □, *QED*, or ∴ goes at the very end.)

In your proofs, you may use

- De Morgan's laws
- contrapositive
- conditional to or
- commutativity of and/or
- associativity of and/or
- negation of both sides of equivalent

Strategies to try for proving equivalent statements:

Make a truth table if allowed.

Go from the left hand side and work your way to the right hand side.

Play around with the statement to be proved. Once you get something you know is true, work backwards.

What's wrong with the format of the following proof of #9c (pg 19)?

$P \Rightarrow (Q \wedge R) \quad \Leftrightarrow \quad (\sim Q \vee \sim R) \Rightarrow \sim P \qquad \textit{Given}$

$\sim P \vee (Q \wedge R) \quad \Leftrightarrow \quad \sim(\sim Q \vee \sim R) \vee \sim P \qquad \textit{Theorem } 1.2a$

$\sim P \vee (Q \wedge R) \quad \Leftrightarrow \quad (Q \wedge R) \vee \sim P \qquad \textit{De Morgan's Laws}$

$\sim P \vee (Q \wedge R) \quad \Leftrightarrow \quad \sim P \vee (Q \wedge R) \qquad \textit{Commutatitivy of } \vee$

Changes from and additions to the book:

- Theorems often use *let, for…we have, suppose,* and *given…we have* to mean *if …then*.
- $A \wedge B \Leftrightarrow B \wedge A$ (recognize this?)
- $A \vee B \Leftrightarrow B \vee A$ (recognize this?)
- $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$ (recognize this?)
- $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ (recognize this?)
- $(A \Leftrightarrow B) \Leftrightarrow (\sim A \Leftrightarrow \sim B)$ (you may negate both sides of equivalent statements)

Lecture 3: Quantifiers (1.3 Part I) Supplement

Classify the quantified statements as True or False.

- All natural numbers are integers.
- No irrational numbers are rational numbers.
- Some real numbers are complex numbers.

Quantified statement	Equivalent statement	Negation
All P are Q.		
No P are Q.		
Some P are Q.		
Some P are not Q.		

Notation (we use slightly different notation than the book)

Read	Symbol	Quantifier
there exists (for some, there is)		
for every (for all, for any, for each)		
there exists a unique (there is one and only one, there is exactly one)		NA
such that (satisfying, with the property)		NA
element of (belongs to, in)		NA

Write in words:

- x is even $\Leftrightarrow \exists k \in Z \ s.t. \ x = 2k$.

- $\forall t \in (1, \infty), \ln t > 0$.

Write symbolically:

- For each x there exists a y such that $\tan y = x$.

- There exists a unique function $f(x)$ such that for every real number x, $f(x) = x$. (recognize this?)

Write in words

- Given that a function f is continuous on $[a, b]$ and differentiable on (a, b),
  $\exists c \in (a,b) \ s.t. \ f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (recognize this?)

- $\forall a, b \in \Re, a = 0 \vee b = 0 \Rightarrow ab = 0$. (recognize this?)

Write symbolically

- A number x is odd iff for some integer k, $x = 2k + 1$.

- For all real numbers r and s, if $r + s < r$, then $s < 0$.

Changes from and additions to the book:

- We define **open sentence** as *a statement containing a variable*.
- The book uses the notation $(\forall x)P(x)$ to denote *for every x, P(x)*. We use the notation $\forall x, P(x)$. The book uses the notation $(\exists x)P(x)$ to denote *there exists an x such that P(x)*. We use the notation $\exists x \ s.t. \ P(x)$. There are many notations out there. If you're interested, http://en.wikipedia.org/wiki/Quantifier#Notation_for_quantifiers has a nice list.

Additional resource used in creating this lecture: *Thinking Mathematically* by Robert Blitzer

Lecture 4: Quantifiers, Direct Proofs (1.3 Part II, 1.4 Part I) Supplement

Statement (symbolic)	Statement in words using some or all.	Negation in words using some or all.	Negation (symbolic)
$\forall x, P(x)$.			
$\exists x$ s.t. $P(x)$.			
$\forall x, \exists y$ s.t. $P(x,y)$.			
$\exists x$ s.t. $\forall y, P(x,y)$.			

Give a negation.

- $\forall q \in Q$, the decimal representation of q repeats or terminates.

- $\exists z$ s.t. $\forall y$, $zy = 0$. (recognize this?)

s

Give a negation (board)

- $\exists t$ s.t. $e^t = 0$.

- $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$. (recognize this?)

- There exists a unique woman such that Romeo loves the woman.

An n x n matrix A is invertible iff $\forall b \in R^n, \exists x \in R^n$ s.t. $Ax = b$. How can I show a matrix is not invertible?

Give the universe for which the statement is true. Choose from $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

$\exists y$ s.t. $2y^3 - y^2 + 2y - 1 = 0$.

Classify the statements as true or false. If a statement involving $\forall$ (all) is false, give a counterexample. If a statement involving $\exists$ (some) is true, give an example.

Statement	Universe	T/F	Counterexample or example (if applicable)
$\forall p, q$, $p - q$ is in the universe	$\mathbb{R}$		
$\forall p, q$, $p - q$ is in the universe	$\mathbb{N}$		
$\exists$ quadrilateral X which is a square	rhombuses		
$\exists$ quadrilateral X which is a square	trapezoids		

Typo in the book: pg 29 $\sim(P \wedge Q) \Leftrightarrow \sim P \wedge \sim Q$ should be $\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$.

Lecture 5: Direct Proofs, Exhaustion, Working Backwards, Previous Results (1.4 Part II, 1.6 Part I)
Supplement

The **rational root test** states that a rational root of a polynomial with integer coefficients and a nonzero constant term, must be in the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

List possible rational roots of $p(x) = 2x^4 + 3x^3 + 4x^2 + 9x - 6$.

Prove there exists a real number *r* such that *p(r) = 0.*

The infamous ε-δ proofs from calculus are construction proofs. $f : X \to Y$ is **continuous** at $a \in X$ iff for every $\varepsilon > 0$, there exists a $\delta > o$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Prove the following functions are continuous at *a*.

a. $f(x) = -x+3$, $a = 2$
b. (Board) $f(x) = \sqrt{x}$, $a = 0$

Suggestions for ε-δ proofs.

Scratchwork:
Plug in everything you can into $|x - a|$ and $|f(x) - f(a)| < \varepsilon$.
Manipulate $|f(x) - f(a)| < \varepsilon$ to look like $|x - a| <$ something. That something will be your δ.

Proof: Let ε > 0. Let δ = … Then $|x - a| < \delta \Leftrightarrow$ … (work backwards until you get $|f(x) - f(a)| < \varepsilon$) We conclude that *f* is continuous at *a*. ■

A limit at positive infinity is defined by $\lim_{x \to +\infty} f(x) = L$ iff $\forall \varepsilon > 0, \exists N > 0 \; s.t. \; |f(x) - L| < \varepsilon$ whenever $x > N$.

Prove $\lim_{x \to +\infty} \frac{1}{x^2} = 0$.