Math 4410 Advanced Calculus II

HW 1
Due Wednesday, January 11, 2006, by 5pm.

1. What are some situations where a function is approximated by a sequence of functions?
2. Use the Taylor series of sin(x) to give several approximations of sin(5°). Compare your approximations to the actual value. Be sure to show your work.
3. Suppose $f(x) \equiv \lim_{n\to\infty} f_n(x)$.
   a. If $f_n(x)$ is continuous $\forall n$, is $f(x)$ continuous?
   b. If $f_n(x)$ is integrable $\forall n$, is $f(x)$ integrable?
   c. If $f_n(x)$ is differentiable $\forall n$, is $f(x)$ differentiable?
   d. If $f(x)$ is continuous, is $f_n(x)$ continuous $\forall n$?

   If the answer is yes, give a theorem to support your answer. Otherwise, give a counterexample.
4. Explain in your own words the difference between pointwise convergence and uniform convergence.
5. How do you prove a sequence of functions does not converge uniformly?
6. Let $f_n(x) = x^n$ and $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$. The domain is [0,1].
   a. Graph f_1, f_2, f_3.
   b. Show that f_n converges to f pointwise.
   c. Prove that f_n does not converge to f uniformly.
7. Given that $g_n(x) = (1-x)x^n$ converges uniformly to 0 on [0,1], find $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx$. Justify your answer.
8. Do 8.2 #6.
9. Why do we care about uniform convergence?