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The λ–Design Conjecture: A Survey

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Abstract:

Let $\mathcal{D}$ be a symmetric (v, k, μ)–design. Fix a block C in $\mathcal{D}$, and form the Ryser–Woodall complementation $\mathcal{E}$ of $\mathcal{D}$ with respect to C. Then $\mathcal{E}$ is a λ–design with $\lambda = k - \mu$. The λ–design conjecture states that all λ–designs can be constructed in this fashion. This paper is a survey of the current (2006) status of this conjecture.

(Please do **not** include this page in the conference proceedings.)

THE λ–DESIGN CONJECTURE: A SURVEY

DEREK W. HEIN

Abstract. Let $\mathcal{D}$ be a symmetric (v, k, μ)–design. Fix a block C in $\mathcal{D}$, and form the Ryser–Woodall complementation $\mathcal{E}$ of $\mathcal{D}$ with respect to C. Then $\mathcal{E}$ is a λ–design with $\lambda = k - \mu$. The λ–design conjecture states that all λ–designs can be constructed in this fashion. This paper is a survey of the current (2006) status of this conjecture.

1. Introduction

The λ–design conjecture has its roots in papers published by H. J. Ryser [21] in 1968 and by D. R. Woodall [33] in 1970, almost forty years ago. In these papers, Ryser and Woodall introduced a new type of combinatorial object called a *square λ–linked design*. This object is the dual of what we call a *λ–design*. (Note that R. G. Stanton [28] calls these objects *Ryser designs*; see also [15].)

Definition 1.1. Let v and λ be fixed positive integers with $0 < \lambda < v$. A *λ–design* $\mathcal{E}$ on v points is a pair $(X, \mathcal{B})$, where X is a set of *points* and $\mathcal{B}$ is a collection of subsets of X (*blocks*) such that

(1) $|X| = |\mathcal{B}| = v$;
(2) $|A \cap B| = \lambda$ for any distinct blocks A and B in $\mathcal{B}$;
(3) $|B| > \lambda$ for any block B in $\mathcal{B}$;
(4) There are blocks A and B in $\mathcal{B}$ such that $|A| \neq |B|$.

Remark 1.2. We note that condition 4 distinguishes a λ–design from a symmetric design [1]. Additionally, condition 3 has several implications: it disallows repeated blocks, and it disallows any block from being contained in another block.

Every λ–design $\mathcal{E}$ on v points has two integers r and r^* (with $r > 1$ and $r^* > 1$) associated with it (called *replication numbers*) such that each point in $\mathcal{E}$ occurs on either r or r^* blocks and $r + r^* = v + 1$. (These integers are very useful in the analysis of λ–designs.) We restate this in Theorem 4.1, and omit the proof.

Remark 1.3. It is interesting to note that from the definition of a λ–design $\mathcal{E}$, nothing is specified about *how many* block sizes $\mathcal{E}$ may have. Nevertheless, the result stated above says that $\mathcal{E}$ has exactly two replication numbers. This is indeed a rather striking result!

All known examples of λ–designs can be described by the following construction that H. J. Ryser [21] gave. Let $\mathcal{D} = (X, \mathcal{A})$ be a symmetric $(v, k, k - \lambda)$–design with $k \neq 2\lambda$. Let A be a fixed block of $\mathcal{D}$. Form the collection $\mathcal{B} = \{A\} \cup \{A \triangle B \colon B \in \mathcal{A}, B \neq A\}$, where $A \triangle B$ denotes the usual symmetric difference of A and B. Then $\mathcal{E} = (X, \mathcal{B})$ is a λ–design on v points. We state this in more detail (and omit the proof) in Theorem 4.2.

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Definition 1.4. Any λ–design $\mathcal{E}$ obtained in this fashion from a symmetric design $\mathcal{D}$ is called a *type–1* λ–design, and we call the λ–design $\mathcal{E}$ formed from the above process the *Ryser–Woodall complementation* of the design $\mathcal{D}$.

H. J. Ryser [21] and D. R. Woodall [33] conjectured that any λ–design can be constructed by this particular method. The central problem in the area of λ–designs is the *λ–design conjecture*:

Conjecture 1.5 (The λ–Design Conjecture). *Every λ–design is a type–1 design.*

2. Old Results

We review some results on λ–designs. H. J. Ryser [21] noted in 1968 that the result of N. G. DeBruijn and P. Erdős [6] from 1948 solved the λ–design conjecture for the case $\lambda = 1$. Ryser then solved the λ–design conjecture for the case $\lambda = 2$. W. G. Bridges and E. S. Kramer [4] showed in 1970 the truthfulness of the λ–design conjecture for $\lambda = 3$. Also, W. G. Bridges [3] showed in 1970 the truthfulness of the λ–design conjecture for $\lambda = 4$.

In 1969, E. S. Kramer [18] wrote his doctoral dissertation in the area of λ–designs. In 1974, he published a paper [19] wherein he proved the truthfulness of the λ–design conjecture for $\lambda = 5$ and gave new proofs of the λ–design conjecture for $\lambda = 2, 3, 4$. He also mentions that in his dissertation, these results were extended to all $\lambda \leq 9$.

In 1976, a very important paper in the area of λ–designs was published by N. M. Singhi and S. S. Shrikhande [27]. They showed the truthfulness of the λ–design conjecture for λ a prime. In 1984 S. S. Shrikhande and N. M. Singhi [26] published a more refined proof of their 1976 result, as well as another useful result. This result will be presented in Theorem 4.7.

In 1989 Á. Seress [25] published a highly technical paper wherein he gave a number of new structural conditions sufficient to imply the truthfulness of the λ–design conjecture. In 1990 Á. Seress [24] showed the truthfulness of the λ–design conjecture for $\lambda = 10$. In 2001, Á. Seress [23] showed the truthfulness of the λ–design conjecture for $\lambda = 2p$, where p is a prime. This result extended the result of W. G. Bridges and T. Tsaur [5] in the $\lambda = 14$ case. (Other structural conditions are given in [2], [14] and [35].)

Definition 1.1 of a λ–design given in the present paper was originally presented by Vijayan [30] in 1992. It was also shown in that paper that in order to prove the λ–design conjecture, it is enough to show that each λ–design (with block size at most 2λ) has a point, all blocks through which have the same size.

In 1995, I. Weisz [32] (a student of Á. Seress) wrote his doctoral dissertation in the area of λ–designs. Therein he proved the truthfulness of the λ–design conjecture for $\lambda \leq 34$, a result that covered many previous results.

3. New Results

In 1996, papers by Y. J. Ionin and M. S. Shrikhande [16, 17] appeared, wherein they developed a new approach to the λ–design conjecture that led to new results for certain parameter values. They also gave an alternate proof of the celebrated non–uniform Fisher Inequality [20].

Y. J. Ionin and M. S. Shrikhande [17] viewed λ–designs as extremal objects in set theory. Let $\mathcal{B}$ denote a collection of subsets (blocks) of $[v] = \{1, 2, \ldots, v\}$ such that any two blocks intersect in

$\lambda(\geq 1)$ points. If $|\mathcal{B}| = v$, then $\mathcal{B}$ is the block set of a symmetric (v, k, μ)–design or of a λ–design (with two obvious exceptions addressed by DeBruijn and Erdős [6]).

Y. J. Ionin and M. S. Shrikhande [17] also went on to explore the truthfulness of the λ–design conjecture from a different perspective. Their results imply the truthfulness of the conjecture for certain values of v rather than for λ. If $\mathcal{B}$ is a λ–design on v points, then the point set $[v]$ is obviously partitioned into subsets E and E^*, where E is the set of points $i \in [v]$ such that the replication number of i is r and E^* is the set of points $j \in [v]$ such that the replication number of j is r^*. They also set $e = |E|$ and $e^* = |E^*|$, so that $e + e^* = v$. It is seen that e and e^* are integers and that $e > 0$ and $e^* > 0$.

For any block $B \in \mathcal{B}$, Y. J. Ionin and M. S. Shrikhande [17] defined $\tau_B = |B \cap E|$ and $\tau_B^* = |B \cap E^*|$, so that $\tau_B + \tau_B^* = |B|$. They observed that if $r > r^*$, then $\tau_B < \lambda < \tau_B^*$ or $\tau_B > \lambda > \tau_B^*$ depending on whether $|B| > 2\lambda$ or $|B| < 2\lambda$ respectively. They also showed that $\tau_B = \lambda = \tau_B^*$ if and only if $|B| = 2\lambda$.

For any λ–design $\mathcal{E} = (X, \mathcal{B})$, fix a block A. By two–way counting the ordered pairs (x, B) (where $x \in X$, $B \in \mathcal{B}$, $B \neq A$, and $x \in A \cap B$) we obtain the formula

$$\tau_B(r-1) + \tau_B^*(r^*-1) = \lambda(v-1)$$

for all blocks $B \in \mathcal{B}$. By replacing r^* by $v + 1 - r$, e^* by $v - e$ and τ_B^* by $|B| - \tau_B$, Y. J. Ionin and M. S. Shrikhande [17] showed by routine manipulations that for all blocks $B \in \mathcal{B}$

$$(r-1)(|B| - 2\tau_B) = (v-1)(|B| - \tau_B - \lambda)$$

Y. J. Ionin and M. S. Shrikhande [17] explored the λ–design conjecture further by defining g to be the greatest common divisor of $r-1$ and r^*-1; that is, $g = \gcd(r-1, r^*-1)$. Since $r + r^* = v + 1$, it is seen that $g = \gcd(r-1, v-1)$ and $g = \gcd(r^*-1, v-1)$.

Y. J. Ionin and M. S. Shrikhande [16, 17] proved that any λ–design with $g \in \{1, 2, 3, 4\}$ is a type–1 λ–design. D. W. Hein and Y. J. Ionin [13] proved that any λ–design with $g = 5$ is a type–1 λ–design. N. C. Fiala [7, 8, 9] proved that any λ–design with $g \in \{6, 7, 8\}$ is a type–1 λ–design.

The results of Y. J. Ionin and M. S. Shrikhande [16, 17] imply that the λ–design conjecture is true for any λ–design on $p+1$, $2p+1$, $3p+1$, or $4p+1$ points (where p is a prime). The results of D. W. Hein and Y. J. Ionin [13] imply that the λ–design conjecture is true for any λ–design on $5p+1$ points (where p is a prime not congruent to 2 or 8 (mod 15)). The results of N. C. Fiala [7, 8] imply that the λ–design conjecture is true for any λ–design on $6p + 1$ points (where p is a prime) or $8p + 1$ points (where p is a prime not congruent to 5 or 11 (mod 24)).

Additionally, N. C. Fiala [10, 11] has developed some feasibility criteria for the existence of λ–designs with two block sizes. These criteria are in the form of integrality conditions, equations, inequalities and Diophantine equations involving various parameters of the designs. These criteria are used to prove the λ–design conjecture for all λ–designs (with two block sizes) with $\lambda < 150$, a result that greatly extends the range of formerly know results.

It should be noted that in the recent book [15] of Y. J. Ionin and M. S. Shrikhande, the last chapter (devoted to λ–designs) covers in detail essentially all known results about λ–designs.

4. Other Results

We review some old results on λ–designs as well as some recent results. Both H. J. Ryser [21, Theorem 1.1] and D. R. Woodall [33, Theorem 2] independently proved the following result:

Theorem 4.1. *In any λ–design $\mathcal{E}$ on v points there exist distinct integers r and r^* (with $r > 1$ and $r^* > 1$) such that any point of $\mathcal{E}$ occurs in either r or r^* blocks and $r + r^* = v + 1$.*

We mention the following general result, with proof left to the reader:

Theorem 4.2. *Let V be a collection of nonempty subsets of an n–set S ($n \geq 1$) with the property that for any two distinct elements A and B of V, $|A \cap B| = \lambda$. Fix one element C of V and form the Ryser–Woodall complementation of V with respect to C. Then, we will have a collection of subsets of S such that for any two distinct elements A and B of this new collection, $|A \cap B| = |C| - \lambda$.*

Remark 4.3. Thus we see that the Ryser–Woodall complementation of a λ–design $\mathcal{E}$ with respect to some block $A \in \mathcal{E}$ is either a symmetric design or a $(|A| - \lambda)$–design.

We now quote a useful result from Y. J. Ionin and M. S. Shrikhande [17]. We have previously noted in Theorem 4.2 and in Remark 4.3 that (under some restrictions) the Ryser–Woodall complementation $\mathcal{E}'$ of a λ–design $\mathcal{E}$ with respect to the block A is again a $(|A| - \lambda)$–design. We denote the new parameters of $\mathcal{E}'$ by $\lambda(A)$, $e(A)$, etc.

Theorem 4.4. *Let $\mathcal{E}$ be a λ–design on v points and let A be a fixed block in $\mathcal{E}$. We denote by $\mathcal{E}(A) = \{A\} \cup \{A \cap B \colon B \in \mathcal{E}, B \neq A\}$ the Ryser–Woodall complementation of $\mathcal{E}$. Then*

(1) $\mathcal{E} = \mathcal{E}(A)(A)$
(2) *If $A \neq E$ and $A \neq E^*$, then $\mathcal{E}(A)$ is a $(|A| - \lambda)$–design, $\mathcal{E}(A)$ has the same replication numbers r and r^* as $\mathcal{E}$. Also, if $\mathcal{E}$ is type–1, then so is $\mathcal{E}(A)$.*
(3) *If $A = E$ or $A = E^*$, then $\mathcal{E}(A)$ is the block set of a symmetric $(v, |A|, |A| - \lambda)$ design.*

We also quote the following result (H. J. Ryser [21], D. R. Woodall [33]):

Theorem 4.5. *Let $\mathcal{E} = (X, \mathcal{B})$ be a λ–design on v points with replication numbers r and r^*. Then*

$$\frac{1}{\lambda} + \sum_{B \in \mathcal{B}} \frac{1}{|B| - \lambda} = \frac{(v-1)^2}{(r-1)(r^*-1)}$$

D. R. Woodall [34, Theorem 3] stated a theorem that implies the following result (a proof can be found in Á. Seress [25]):

Theorem 4.6 (Woodall Divisibility Condition)**.** *A λ–design on v points with replication numbers r and r^* is type–1 if and only if $\dfrac{r(r-1)}{v-1}$ or $\dfrac{r^*(r^*-1)}{v-1}$ is a (positive) integer.*

The following theorem was proved by S. S. Shrikhande and N. M. Singhi [26, Theorem 4.6]. We note that in their statement of this Theorem, there is a typographical error where the statement "$(\rho, c - d) = 1$" should appear as "$(\lambda, c - d) = 1$" (meaning "$gcd(\lambda, c - d) = 1$").

Theorem 4.7. *Let $\mathcal{E}$ be a λ–design with distinct replication numbers r and r^* guaranteed by Theorem 4.1. Without loss of generality let $r > r^*$ and let $\rho = \dfrac{r-1}{r^*-1} > 1$. If $\rho = \dfrac{c}{d}$ with c and d relatively prime, and if λ and $c - d$ are relatively prime, then $\mathcal{E}$ is a type–1 λ–design.*

We include a few results contained in Y. J. Ionin and M. S. Shrikhande [17]:

Lemma 4.8. *If a λ–design $\mathcal{E}$ on v points has a block of cardinality $v-1$, then $\lambda = 1$.*

Remark 4.9. We note that this lemma implies that if a λ–design $\mathcal{E}$ on v points has a block of cardinality $v-1$, then $\mathcal{E}$ is a type–1 λ–design (from DeBruijn and Erdős [6]).

In all, there are many cases of the λ–design conjecture that have already been proved:

Theorem 4.10. *The λ–design conjecture is true for all positive integers $\lambda \leq 34$. In addition, the λ–design conjecture is true (for λ–designs with two block sizes) for all positive integers $\lambda < 150$.*

The first part of Theorem 4.10 was proved by I. Weisz [32] in his 1995 doctoral dissertation. The second part of Theorem 4.10 was proved by N. C. Fiala [10, 11].

Remark 4.11. We note that the result of N. M. Singhi and S. S. Shrikhande [27] also covers all other prime λ, and that the result of Á. Seress [23] covers all other values of λ equal to twice a prime. We also note that the case $\lambda = 150$ is still the smallest unresolved case.

The following theorem was proved by Y. J. Ionin and M. S. Shrikhande [16, 17], D. W. Hein and Y. J. Ionin [13] and N. C. Fiala [7, 8]:

Theorem 4.12. *Any λ–design with $g \in \{1,2,3,4,5,6,7,8\}$ is a type–1 λ–design. Furthermore, the λ–design conjecture is true for all λ–designs on v points, where $v \in \{p+1, 2p+1, 3p+1, 4p+1, 6p+1\}$ (where p is a prime), $v = 5p+1$ (where p is a prime not congruent to 2 or $8 \pmod{15}$) or $v = 8p+1$ (where p is a prime not congruent to 5 or $11 \pmod{24}$).*

5. Conclusion

N. C. Fiala [9, 10, 11] leaves several open questions: Since the λ–design conjecture is true for $g = 7$, what can be said about λ–designs on $v = 7p+1$ points? What about λ–designs with exactly two block sizes (in general)? How can modern technology be more effectively used to further the resolution of the conjecture?

The work of D. W. Hein and Y. J. Ionin [13] left the cases of $p \equiv 2$ or $8 \pmod{15}$ open (in the $g = 5$ result). Similarly, N. C. Fiala [8] left the cases of $p \equiv 5$ or $11 \pmod{24}$ open (in the $g = 8$ result). Hence, there remains work to be done to prove these small pieces of the λ–design conjecture to gain "full" results in these cases.

Remark 5.1. We note that further assumptions on the equivalence of p modulo other primes may narrow the remaining undecided cases, but will never result in a full resolution of the λ–design conjecture in the cases $v = 5p+1$ or $v = 8p+1$.

Thus we see that the λ–design conjecture has a rich history, with a variety of clever techniques from many diligent researchers. Nevertheless, the conjecture is still unsettled in general, and remains an open problem today.

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