A Survey of

Sarvate–Beam Designs

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Abstract

A *Sarvate–Beam design* is a set V of v elements and a collection of k–subsets of V (called *blocks*) such that each distinct pair of elements in V occurs in i blocks, for every i in the list $1, 2, \ldots, \binom{v}{2}$. In this talk, we discuss recent results on Sarvate–Beam designs.

Chapter 0

Introduction

Example 0.1.

Suppose that we wish to design an experiment with 9 drugs and 12 rats such that

1. each drug is tested on the same number of rats;
2. each rat has the same number of drugs tested on it;
3. any two drugs occur in exactly one rat (for reaction comparison purposes)

Can this be done?

Yes, in the following fashion:

Arbitrarily label the drugs $\{1, \ldots, 9\}$ and the rats $\{1, \ldots, 12\}$. Then let

$R_1 = \{1, 2, 3\}$, $R_2 = \{1, 4, 5\}$,

$R_3 = \{1, 6, 7\}$, $R_4 = \{1, 8, 9\}$,

$R_5 = \{2, 4, 7\}$, $R_6 = \{2, 5, 8\}$,

$R_7 = \{2, 6, 9\}$, $R_8 = \{3, 4, 9\}$,

$R_9 = \{3, 5, 6\}$, $R_{10} = \{3, 7, 8\}$,

$R_{11} = \{4, 6, 8\}$, $R_{12} = \{5, 7, 9\}$.

Definition 0.2.

A (v, k, λ)–*design* is a pair $(X, \mathcal{B})$, where X is a v–set of elements called *points* and $\mathcal{B}$ is a collection of subsets of X called *blocks* that satisfy

1. each block has the same cardinality k;

2. any two points of X occur in exactly λ blocks.

Remark 0.3.

In a (v, k, λ)–design, each point occurs in a constant number of blocks. We denote this by r and call it the *replication number* of the design. ▼

Remark 0.4.

What we have defined as a (v, k, λ)–design is customarily called a "balanced incomplete block design" (or BIBD) in the literature. ▼

Definition 0.5.

A (v, k, λ)–design is called *symmetric* if there are v blocks in the design.

Example 0.6.

Consider the *Fano Plane*. It is a symmetric $(7, 3, 1)$–design with point set

$$X = \{\,1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\,\}$$

and block set

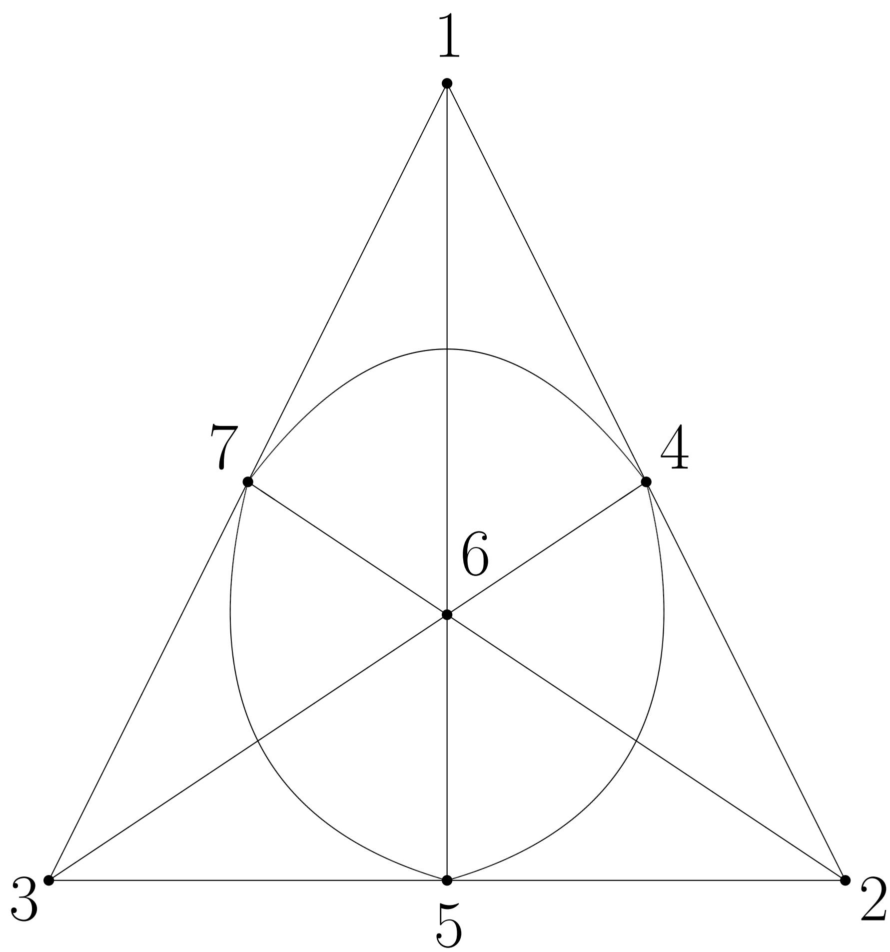
$$\mathcal{B} = \{\, B_1,\ B_2,\ B_3,\ B_4,\ B_5,\ B_6,\ B_7\,\}$$

$$= \{\,\{\,1,\ 2,\ 4\},\ \{\,2,\ 3,\ 5\,\},\ \{\,3,\ 4,\ 6\,\},$$

$$\{\,4,\ 5,\ 7\,\},\ \{\,5,\ 6,\ 1\,\},\ \{\,6,\ 7,\ 2\,\},$$

$$\{\,7,\ 1,\ 3\,\}\,\}$$

It can be represented by the following diagram:



It can also be represented by the $b \times v$ *incidence matrix*:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \blacktriangledown$$

Chapter 1

Sarvate–Beam Designs

The present area under consideration has its roots in papers published since 2007 by D. Sarvate, W. Beam, R. Stanton and others.

In these papers, Sarvate and Beam [7] introduced a new type of combinatorial object called an *adesign*.

Definition 1.1.

An *adesign* $AD(v,k)$ is a set V of v elements and a collection of k–subsets of V (called *blocks*) such that each distinct pair of elements in V has a different *frequency* (that is, it occurs in a different number of blocks).

Definition 1.2.

A *strict* adesign $SAD(v, k)$ is an adesign such that each frequency occurs exactly once from the list 1, 2, $\ldots$, $\binom{v}{2}$.

Remark 1.3.

Definition 1.1 was given by Sarvate and Beam, although the definition of the term "frequency" was given by Dukes [3]. We note that the frequency condition distinguishes an adesign from a balanced incomplete block design (BIBD).

Definition 1.4.

An $aPBD(v, K)$ is a set V of v elements and a collection of subsets of V such that each distinct pair of elements in V has a different frequency and the size of any block is in K.

Remark 1.5.

Definition 1.4 was also given by Sarvate and Beam [7].

Definition 1.6.

Of course, a *strict* aPBD $SaPBD(v, K)$ is a set V of v elements and a collection of subsets of V such that each distinct pair of elements in V has a different frequency occurring exactly once from the list 1, 2, $\ldots$, $\binom{v}{2}$, and the size of any block is in K.

Remark 1.7.

This term was coined in [1].

The definition of "strict adesign" was renamed by Stanton [9] as a *Sarvate–Beam design* (or *SB design*).

Stanton [9] also introduced the term "SB Triple System" [referring to a $SAD(v, 3)$]. He generalized his terminology in [11] to a "SB Quad System" [which is a $SAD(v, 4)$].

Dukes [3] improved the notation, and labels it $SB(v, k)$. Thus, we have $SB(v, 3)$ [for SB Triple Systems] and $SB(v, 4)$ [for SB Quad Systems]. The notation can further be used to symbolize a $SaPBD(v, K)$ by $SB(v, K)$.

Chapter 2

Preliminaries

(Using Stanton's [9] notation,) several examples of SB designs have been provided:

Example 2.1 (Sarvate, Beam [7]).

$124 + 2(134) + 4(234)$ is a $SB(4, 3)$.

Remark 2.2.

This was proved by Stanton [9] to be unique up to isomorphism.

Example 2.3 (Sarvate, Beam [7]).

$124+2(135)+2(145)+234+4(235)+5(245)+7(345)$ is an $AD(5,3)$.

Remark 2.4.

Note that it is not a strict adesign. Hence, it does not contradict Theorem 5.1.

Example 2.5 (Stanton [13]).

$$10(12) + 135 + 3(145) + 2(234) + 4(235) + 5(245)$$

is a $SB(5, \{2, 3\})$.

Bradford [1] produced the following examples:

Example 2.6.

$1234 + 2(135) + 3(145) + 234 + 4(235) + 6(245) + 5(345)$ is a (non–strict) $aPBD(5, \{3, 4\})$.

Example 2.7.

$1234 + 134 + 145 + 2(146) + 2(156) + 3(235) + 2(236) + 3(245) + 3(246) + 3(256) + 4(345) + 4(346) + 5(356) + 5(456)$ is a $SB(6, \{3, 4\})$.

Stanton [9] goes on to make another definition:

Definition 2.8.

An SB design is called *restricted* if only blocks beginning with 1 or 2 (up to isomorphism) are allowable.

Also, Stanton [10] gave an example of a restricted $SB(8, \{2, 3\})$.

Chapter 3

SB Quad Systems

Stanton [11] analyzes SB Quad Systems. He gives the following example:

Example 3.1.

$1234 + 2(1345) + 2(1356) + 5(2345) + 6(2346) + 3(2356) + 2456$ is a restricted $SB(6,4)$.

The following example is given by Sarvate and Beam [8]:

Example 3.2.

$1236 + 1356 + 3(1456) + 2(2346) + 3(2356) + 5(2456) + 8(3456)$ is a (non–strict) $AD(6, 4)$.

Stanton [11] proved the following:

Theorem 3.3.

The only restricted $SB(v, 4)$ *that possibly exist are for* $v \in \{6, 9\}$.

Chapter 4

SB Quad Systems with $v = 6$

In his analysis of restricted SB Quad Systems when $v = 6$, Stanton [11] makes assignments for block frequencies.

He assigns the frequency 1 to the block 1234, a_1 to the block 1345, a_2 to the block 1346, a_3 to the block 1356, a_4 to the block 1456, b_1 to the block 2345, b_2 to the block 2346, b_3 to the block 2356, b_4 to the block 2456 and 0 to all other blocks.

Stanton then labels the frequency of the pair $\{a, b\}$ by $f(ab)$, noting that $f(12) = 1$. Also,

$$f(13) = 1 + a_1 + a_2 + a_3$$

$$f(14) = 1 + a_1 + a_2 + a_4$$

$$f(15) = a_1 + a_3 + a_4$$

$$f(16) = a_2 + a_3 + a_4$$

$$f(23) = 1 + b_1 + b_2 + b_3$$

$$f(24) = 1 + b_1 + b_2 + b_4$$

$$f(25) = b_1 + b_3 + b_4$$

$$f(26) = b_2 + b_3 + b_4$$

$$f(34) = 1 + a_1 + a_2 + b_1 + b_2$$

$$f(35) = a_1 + a_3 + b_1 + b_3$$

$$f(36) = a_2 + a_3 + b_2 + b_3$$

$$f(45) = a_1 + a_4 + b_1 + b_4$$

$$f(46) = a_2 + a_4 + b_2 + b_4$$

$$f(56) = a_3 + a_4 + b_3 + b_4$$

We consider the existence of a restricted $SB(6,4)$ to be equivalent to finding an ordered 9–tuple $(1, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying the above conditions and such that $1 \leq f(ab) \leq 15 = \binom{6}{2}$ with each $f(ab)$ a distinct integer.

Stanton also introduces parameters $A = \sum a_i$ and $B = \sum b_i$ and shows that $A + B = 19$. He further shows that $A \geq 4$. Assuming that $A \leq B$, this leaves 6 cases of possibilities for restricted $SB(6, 4)$:

$(A, B) \in \{(4, 15), (5, 14), (6, 13), (7, 12), (8, 11), (9, 10)\}$.

Each of these cases requires checking large numbers of possible 9–tuples that can produce restricted $SB(6, 4)$.

It is a simple matter to write a computer code to check all of the conditions for each possible 9–tuple.

When we did this, we got the following results:

(A, B)	solutions	nonisomorphic
(4, 15)	16	4
(5, 14)	0	0
(6, 13)	0	0
(7, 12)	0	0
(8, 11)	0	0
(9, 10)	0	0

These solutions correspond to the 9-tuples

$(1,0,1,0,3,4,0,8,3)$, $(1,0,1,0,3,7,0,6,2)$,

$(1,0,2,0,2,6,5,1,3)$ and $(1,1,2,0,1,0,2,8,5)$.

Thus we have shown the following:

Theorem 4.1 (Bradford, Hein and Pace [1])**.**

There are 4 nonisomorphic restricted

$SB(6,4)$.

To extend his analysis to unrestricted SB Quad Systems, Stanton additionally assigns the frequency d_1 to the only other possible block 3456 (keeping the other assignments in the restricted case the same).

Again denoting the frequency of the pair $\{a, b\}$ by $f(ab)$, he notes that

$$f(34) = 1 + a_1 + a_2 + b_1 + b_2 + d_1$$

$$f(35) = a_1 + a_3 + b_1 + b_3 + d_1$$

$$f(36) = a_2 + a_3 + b_2 + b_3 + d_1$$

$$f(45) = a_1 + a_4 + b_1 + b_4 + d_1$$

$$f(46) = a_2 + a_4 + b_2 + b_4 + d_1$$

$$f(56) = a_3 + a_4 + b_3 + b_4 + d_1$$

(with the other frequencies in the restricted case the same).

We consider the existence of an unrestricted $SB(6,4)$ to be equivalent to finding an ordered 10–tuple $(1, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, d_1)$ satisfying the above conditions and such that $1 \leq f(ab) \leq 15$ with each $f(ab)$ a distinct integer.

Stanton also introduced the parameter $D = \sum d_i = d_1$ and noted that $A + B + D = 19$. Retaining the restriction $A \geq 4$ and the assumption that $A \leq B$, this again leaves 6 cases of possibilities for unrestricted $SB(6, 4)$:

$(A, B + D) \in \{(4, 15), (5, 14), (6, 13), (7, 12), (8, 11), (9, 10)\}$.

Each of these cases requires checking large numbers of possible 10–tuples that can produce unrestricted $SB(6,4)$. We generalized the computer code used in the unrestricted case to check all of the conditions for each possible 10–tuple.

When we did this, we got the following results:

$(A, B + D)$	solutions	nonisomorphic
(4, 15)	24	6
(5, 14)	4	1
(6, 13)	0	0
(7, 12)	0	0
(8, 11)	0	0
(9, 10)	0	0

These solutions correspond to the 10-tuples

$(1, 0, 1, 0, 3, 4, 2, 1, 6, 2), (1, 0, 2, 0, 2, 1, 6, 4, 1, 3),$

$(1, 0, 2, 0, 2, 5, 1, 8, 0, 1), (1, 0, 2, 0, 2, 7, 4, 1, 2, 1),$

$(1, 1, 2, 0, 1, 0, 6, 4, 2, 3), (1, 1, 2, 0, 1, 6, 1, 6, 0, 2)$

and $(1, 0, 1, 0, 4, 8, 0, 3, 0, 3)$. Thus we have shown the following:

Theorem 4.2 (Hein [1])**.**

There are 7 nonisomorphic unrestricted $SB(6, 4)$.

Remark 4.3.

We note that there are no restricted $SB(6,4)$ when $A = 5$, yet there exists an unrestricted $SB(6,4)$ when $A = 5$. This was an unexpected yet pleasant discovery!

Theorems 4.1 and 4.2 can be combined into one summary result:

Theorem 4.4 (Bradford, Hein and Pace [1])**.**

There are 11 nonisomorphic $SB(6,4)$*.*

The computer code used to produce this result can be found at the website

`http://www.suu.edu/faculty/hein/professional.html`

We also furnish the results of the code on the website.

It is written in C++, and produces 5 files — 4 are auxiliary files used by the program, and a file called **`u64solns.dat`** with all 44 solutions. (These solutions were checked by hand to produce the nonisomorphic solutions). The total runtime for the code on our local server is 0.366 second.

Chapter 5

SB Triple Systems

Sarvate and Beam [7] gave examples of unrestricted $SB(6,3)$ and $SB(7,3)$.

Stanton [9] gave examples of unrestricted $SB(6,3)$ and $SB(7,3)$, neither of which are isomorphic to Sarvate and Beam's examples.

Stanton [9] also gave examples of restricted $SB(6,3)$ and $SB(7,3)$, neither of which can be isomorphic to any previous examples.

Sarvate and Beam [7] proved the following:

Theorem 5.1.

For a $SB(v,3)$*, it must be true that* $v \equiv 0,1 \pmod 3$*.*

The following is an immediate consequence:

Corollary 5.2 (Bradford and Pace [1])**.**

For a $SB(v, \{3, 4\})$, *it must be true that* $v \equiv 0, 1 \pmod{3}$.

Stanton [12] outlined a general procedure for finding bounds on possible values of v for restricted SB Triple Systems. He proved the following:

Theorem 5.3.

When $k \leq 3$, the only restricted $SB(v, k)$ that possibly exist are for $v \in \{4, 5, 6, 7, 8\}$.

Stanton [13] addressed the $v \equiv 2 \pmod 3$ case for $SB(v, 3)$. He did so by allowing a single pair (12) in the designs. Thus, even though he still calls them "SB Triple Systems", they are technically $SB(v, \{2, 3\})$.

Chapter 6

SB Triple Systems with $v = 6$

Proceeding in a fashion similar to Stanton's approach, all $SB(6,3)$ have been enumerated:

Theorem 6.1 (Hein and Li [5]).

There are 48,843 nonisomorphic restricted

$SB(6,3)$.

The computer code used to produce this result can be found at the website

`http://www.suu.edu/faculty/hein/professional.html`

We also furnish the results of the code on the website.

It is written in C++, and produces 5 files — 4 are auxiliary files used by the program, and a file called **`r63solns.dat`** with all 293,058 solutions. (These solutions were checked by hand [noting a pattern!] to produce the nonisomorphic solutions). The total runtime for the code on our local server is 24 minutes 58.549 seconds.

This result was independently verified by P. C. Li, using Brendan McKay's program **`nauty`**.

Theorem 6.2 (Li [5]).

There are 16,395,407 nonisomorphic unrestricted $SB(6,3)$.

Remark 6.3.

This result was computed by P. C. Li, again using `nauty`.

Theorems 6.1 and 6.2 can be combined into one summary result:

Theorem 6.4 (Hein and Li [5])**.** *There are 16,444,250 nonisomorphic* $SB(6,3)$*.*

These designs are enumerated on the website `http://limba.cs.umanitoba.ca/work.html`

Chapter 7

SB Triple Systems with $v = 5$

Stanton analyzes $SB(5, \{2, 3\})$ with $f(12) = 1$.

Theorem 7.1 (Stanton [13])**.**

There are 20 nonisomorphic $SB(5, \{2, 3\})$ *with* $f(12) = 1$ *(10 restricted, 10 unrestricted).*

He remarks that it is possible for us to have $f(12) \in \{1, 4, 7, 10\}$. These are all different designs, since the frequencies of the pair (12) are different. Hence, we enumerate all $SB(5, \{2, 3\})$ with $f(12) \in \{4, 7, 10\}$.

After proceeding in a fashion similar to the above, we arrive at the following:

Theorem 7.2 (Hein and Li [5]).

There are 11 nonisomorphic $SB(5, \{2, 3\})$ *with* $f(12) = 4$ *(2 restricted, 9 unrestricted).*

There are 17 nonisomorphic $SB(5, \{2, 3\})$ *with* $f(12) = 7$ *(5 restricted, 12 unrestricted).*

There are 16 nonisomorphic $SB(5, \{2, 3\})$ *with* $f(12) = 10$ *(7 restricted, 9 unrestricted).*

Theorems 7.1 and 7.2 can be combined into one summary result:

Theorem 7.3 (Hein and Li [5])**.**

There are 64 nonisomorphic $SB(5, \{2, 3\})$ *(24 restricted, 40 unrestricted).*

The computer code used to produce this result can also be found at the website

`http://www.suu.edu/faculty/hein/professional.html`

We also furnish the results of the code on the website.

It is written in C++, and produces 5 files — 4 are auxiliary files used by the program, and a file called `u53solnst.dat` with all 66 nonisomorphic solutions. The total runtime for the code on our local server is 0.155 second.

Chapter 8

Other Directions

We now consider a generalization:

Definition 8.1.

A t–*SB design* is a collection B of k–subsets of a v–set V such that each t–subset of V occurs a distinct number of times.

Remark 8.2.

Definition 8.1 was given by Sarvate and Beam [8].

There have been non–existence results:

Theorem 8.3 (Sarvate and Beam [8])**.**

For $n > 4$*, a strict* $(n-2) - SB(n, n-1)$

does not exist.

Theorem 8.4 (Chan and Sarvate [2])**.**

For $n > 4$*, a strict* $t - SB(n, n-1)$ *does not*

exist for $2 \leq t \leq n-3$*.*

There have been existence results:

Theorem 8.5 (Sarvate and Beam [8]).

Non–strict $(n-2)-SB(n,n-1)$ *exist for all integers* $n \geq 3$.

Theorem 8.6 (Chan and Sarvate [2]).

Non–strict $2-SB(n,n-1)$ *exist for all integers* $n \geq 3$.

Chapter 9

Open Problems

Stanton [9] states several open questions:

- Find the number of nonisomorphic $SB(7, 3)$. (Stanton [9] remarks that this number "may be too great for easy handling".)

- Find the number of nonisomorphic restricted $SB(7,3)$.
- Find the number of nonisomorphic restricted $SB(8,\{2,3\})$. (Stanton [10] remarks that this number "appears large".)
- Find the number of nonisomorphic $SB(9,4)$. (Stanton [11] remarks that this enumeration "will be cumbersome".)
- Find the number of nonisomorphic restricted $SB(9,4)$.

- Find the number of nonisomorphic $SB(v, 4)$ for $v > 9$. (Stanton [11] remarks that this enumeration "is daunting".)

There are also other non–existence results to obtain:

- Construct all non–strict $t - SB(n, n-1)$.
- Construct all non–strict $t - SB(n, n-2)$.

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