Solid Mechanics: Failure Criteria
Failure Criteria for Ductile Materials

Maximum Shear Stress Criterion

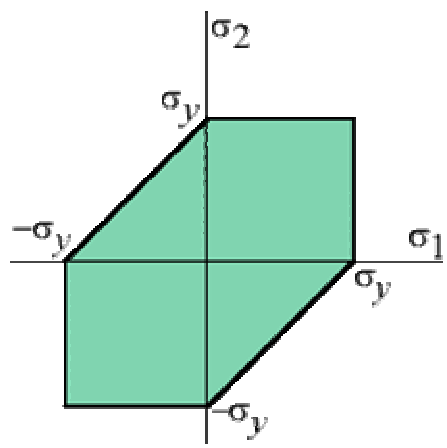
The maximum shear stress criterion, also known as Tresca's or Guest's criterion, is often used to predict the yielding of ductile materials.

Yield in ductile materials is usually caused by the *slippage* of crystal planes along the maximum shear stress surface. Therefore, a given point in the body is considered safe as long as the maximum shear stress at that point is under the yield shear stress σ_y obtained from a uniaxial tensile test.

With respect to 2D stress, the maximum shear stress is related to the difference in the two principal stresses (see Mohr's Circle). Therefore, the criterion requires the principal stress difference, along with the principal stresses themselves, to be less than the yield shear stress,

$$|\sigma_1| \leq \sigma_y, \ |\sigma_2| \leq \sigma_y, \text{ and } |\sigma_1 - \sigma_2| \leq \sigma_y$$

Graphically, the maximum shear stress criterion requires that the two principal stresses be within the green zone indicated below,



Von Mises Criterion

The von Mises Criterion (1913), also known as the maximum distortion energy criterion, octahedral shear stress theory, or Maxwell-Huber-Hencky-von Mises theory, is often used to estimate the yield of ductile materials.

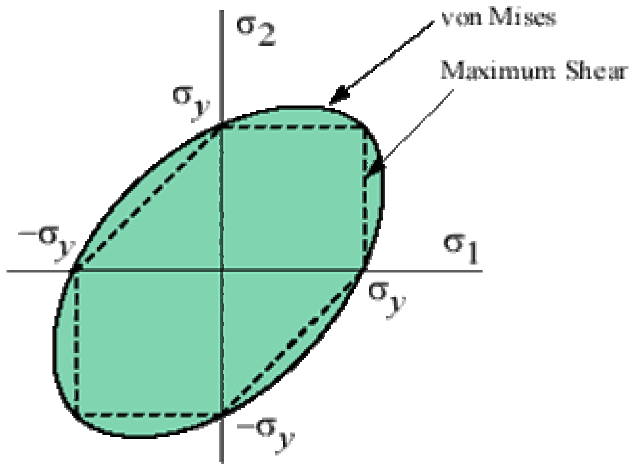
The von Mises criterion states that failure occurs when the energy of distortion reaches the same energy for yield/failure in uniaxial tension. Mathematically, this is expressed as,

$$\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\le\sigma_y{}^2$$

In the cases of plane stress, $\sigma_3 = 0$. The von Mises criterion reduces to,

$$\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2\le\sigma_y^2$$

This equation represents a principal stress ellipse as illustrated in the following figure,



Also shown on the figure is the maximum shear stress criterion (dashed line). This theory is more conservative than the von Mises criterion since it lies inside the von Mises ellipse.

In addition to bounding the principal stresses to prevent ductile failure, the von Mises criterion also gives a reasonable estimation of fatigue failure, especially in cases of repeated tensile and tensile-shear loading

Solid Mechanics: Failure Criteria
Failure Criteria for Brittle Materials

Maximum Normal Stress Criterion

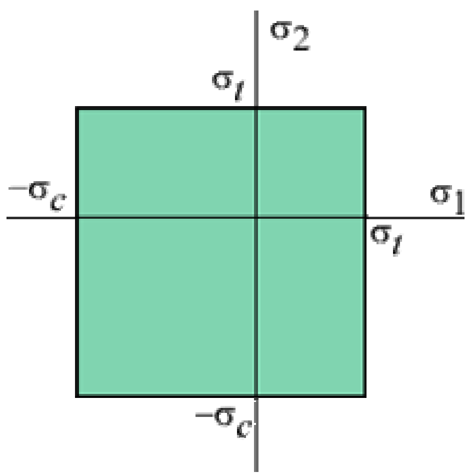
The maximum stress criterion, also known as the normal stress, Coulomb, or Rankine criterion, is often used to predict the failure of brittle materials.

The maximum stress criterion states that failure occurs when the maximum (normal) principal stress reaches either the *uniaxial* tension strength σ_t, or the *uniaxial* compression strength σ_c,

$$-\sigma_c < \{\sigma_1, \sigma_2\} < \sigma_t$$

where σ_1 and σ_2 are the principal stresses for 2D stress.

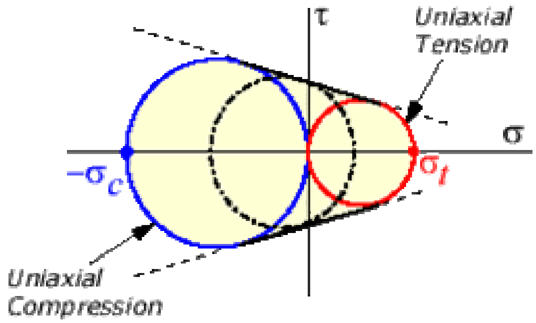
Graphically, the maximum stress criterion requires that the two principal stresses lie within the green zone depicted below,



Mohr's Theory

The Mohr Theory of Failure, also known as the Coulomb-Mohr criterion or internal-friction theory, is based on the famous Mohr's Circle. Mohr's theory is often used in predicting the failure of brittle materials, and is applied to cases of 2D stress.

Mohr's theory suggests that failure occurs when Mohr's Circle at a point in the body exceeds the envelope created by the two Mohr's circles for uniaxial tensile strength and uniaxial compression strength. This envelope is shown in the figure below,



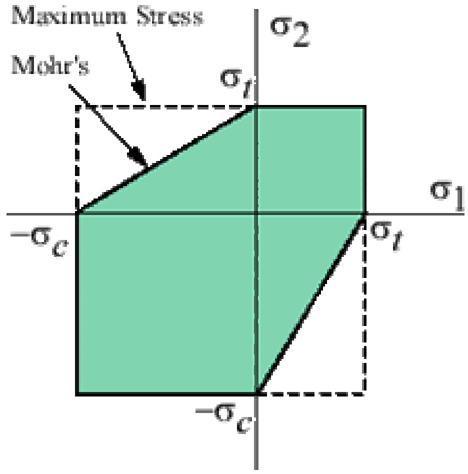
The left circle is for uniaxial compression at the limiting compression stress σ_c of the material. Likewise, the right circle is for uniaxial tension at the limiting tension stress σ_t.

The middle Mohr's Circle on the figure (dash-dot-dash line) represents the maximum allowable stress for an intermediate stress state.

All intermediate stress states fall into one of the four categories in the following table. Each case defines the maximum allowable values for the two principal stresses to avoid failure.

Case	Principal Stresses		Criterion requirements
1	Both in tension	$\sigma_1 > 0,\ \sigma_2 > 0$	$\sigma_1 < \sigma_t,\ \sigma_2 < \sigma_t$
2	Both in compression	$\sigma_1 < 0,\ \sigma_2 < 0$	$\sigma_1 > -\sigma_c,\ \sigma_2 > -\sigma_c$
3	σ_1 in tension, σ_2 in compression	$\sigma_1 > 0,\ \sigma_2 < 0$	$\frac{\sigma_1}{\sigma_t} + \frac{\sigma_2}{-\sigma_c} < 1$
4	σ_1 in compression, σ_2 in tension	$\sigma_1 < 0,\ \sigma_2 > 0$	$\frac{\sigma_1}{-\sigma_c} + \frac{\sigma_2}{\sigma_t} < 1$

Graphically, Mohr's theory requires that the two principal stresses lie within the green zone depicted below,



Also shown on the figure is the maximum stress criterion (dashed line). This theory is less conservative than Mohr's theory since it lies outside Mohr's boundary.

Solid Mechanics: Stress
Principal Stress for the Case of Plane Stress

Principal Directions, Principal Stress

The normal stresses ($\sigma_{x'}$ and $\sigma_{y'}$) and the shear stress ($\tau_{x'y'}$) vary smoothly with respect to the rotation angle θ, in accordance with the coordinate transformation equations. There exist a couple of particular angles where the stresses take on special values.

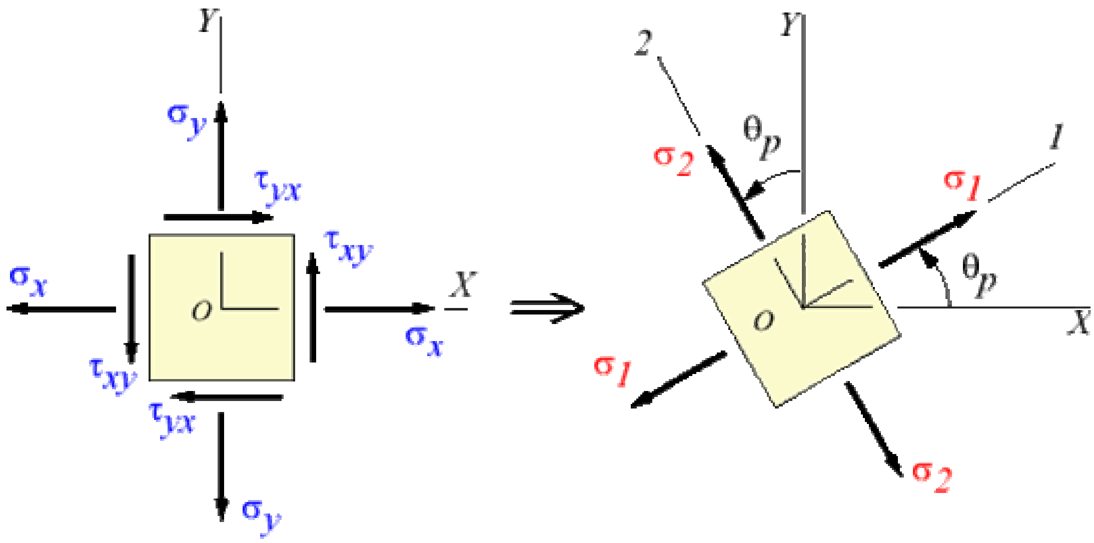
First, there exists an angle θ_p where the shear stress $\tau_{x'y'}$ becomes zero. That angle is found by setting $\tau_{x'y'}$ to zero in the above shear transformation equation and solving for θ (set equal to θ_p). The result is,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The angle θ_p defines the *principal directions* where the only stresses are normal stresses. These stresses are called *principal stresses* and are found from the original stresses (expressed in the *x*,*y*,*z* directions) via,

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The transformation to the principal directions can be illustrated as:



Stresses in given coordinate system

Principal stresses

Maximum Shear Stress Direction

Another important angle, θ_s, is where the maximum shear stress occurs. This is found by finding the maximum of the shear stress transformation equation, and solving for θ. The result is,

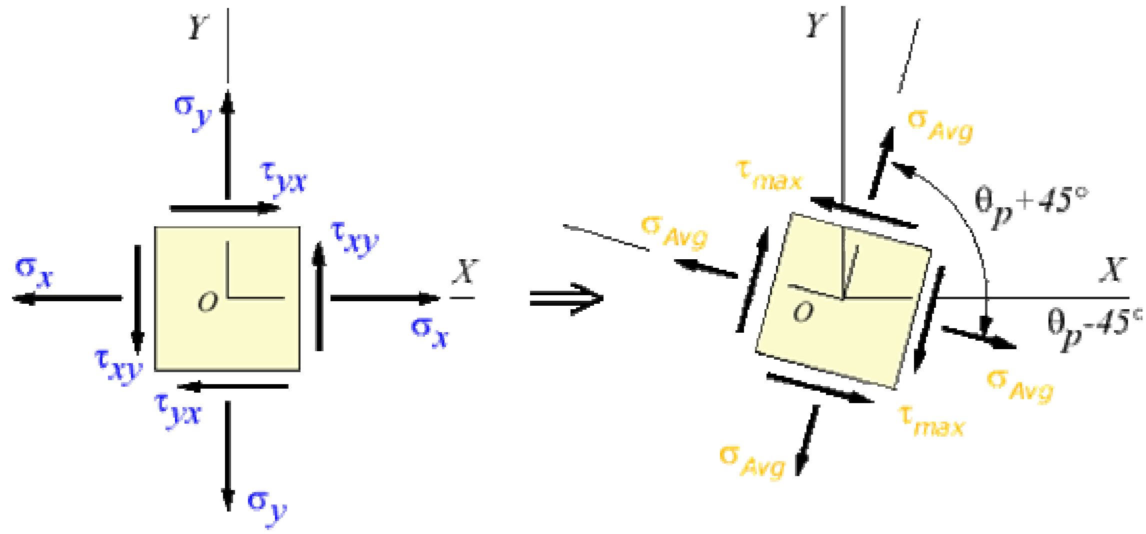
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear stress is equal to one-half the difference between the two principal stresses,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

The transformation to the maximum shear stress direction can be illustrated as:

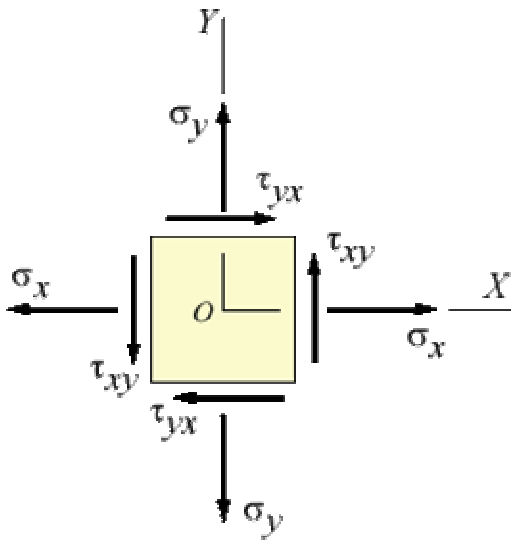


Stresses in given coordinate system

Maximum shear stress

Solid Mechanics: Stress
Plane Stress and Coordinate Transformations

Plane State of Stress



A class of common engineering problems involving stresses in a thin plate or on the free surface of a structural element, such as the surfaces of thin-walled pressure vessels under external or internal pressure, the free surfaces of shafts in torsion and beams under transverse load, have one principal stress that is much smaller than the other two. By assuming that this small principal stress is zero, the three-dimensional stress state can be reduced to two dimensions. Since the remaining two principal stresses lie in a plane, these simplified 2D problems are called **plane stress** problems.

Assume that the negligible principal stress is oriented in the *z*-direction. To reduce the 3D stress matrix to the 2D plane stress matrix, remove all components with *z* subscripts to get,

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

where $\tau_{xy} = \tau_{yx}$ for static equilibrium. The sign convention for positive stress components in plane stress is illustrated in the above figure on the 2D element.

Coordinate Transformations

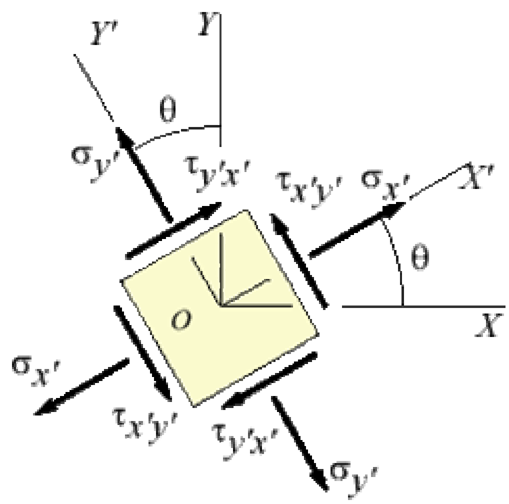
The coordinate directions chosen to analyze a structure are usually based on the shape of the structure. As a result, the direct and shear stress components are associated with these directions. For example, to analyze a bar one almost always directs one of the coordinate directions along the bar's axis.

Nonetheless, stresses in directions that do not line up with the original coordinate set are also important. For example, the failure plane of a brittle shaft under torsion is often at a 45° angle with respect to the shaft's axis. Stress transformation formulas are required to analyze these stresses.

The transformation of stresses with respect to the {*x*,*y*,*z*} coordinates to the stresses with respect to {*x'*,*y'*,*z'*} is performed via the equations,

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \quad\;\; = \sigma_x + \sigma_y - \sigma_{x'} \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

where θ is the rotation angle between the two coordinate sets (positive in the counterclockwise direction). This angle along with the stresses for the {*x'*,*y'*,*z'*} coordinates are shown in the figure below,



Solid Mechanics: Failure Criteria
Techniques for Failure Prevention and Diagnosis

There exist a set of basic techniques for preventing failure in the design stage, and for diagnosing failure in manufacturing and later stages.

In the Design Stage

It is quite commonplace today for design engineers to verify design stresses with finite element (FEA) packages. This is fine and good when FEA is applied appropriately. However, the popularity of finite element analysis can condition engineers to look just for red spots in simulation output, without really understanding the essence or *funda* at play.

By following basic rules of thumb, such danger points can often be anticipated and avoided without total reliance on computer simulation.

Loading Points	Maximum stresses are often located at loading points, supports, joints, or maximum deflection points.
Stress Concentrations	Stress concentrations are usually located near corners, holes, crack tips, boundaries, between layers, and where cross-section areas change rapidly. Sound design avoids rapid changes in material or geometrical properties. For example, when a large hole is removed from a structure, a reinforcement composed of generally no less than the material removed should be added around the opening.
Safety Factors	The addition of safety factors to designs allow engineers to reduce sensitivity to manufacturing defects and to compensate for stress prediction limitations.

In Post-Manufacturing Stages

Despite the best efforts of design and manufacturing engineers, unanticipated failure may occur in parts after design and manufacturing. In order for projects to succeed, these failures must be diagnosed and resolved quickly and effectively. Often, the failure is caused by a singular factor, rather than an involved collection of factors.

Such failures may be caught early in initial quality assurance testing, or later after the part is delivered to the customer.

Induced Stress Concentrations	Stress concentrations may be induced by inadequate manufacturing processes. For example, initial surface imperfections can result from sloppy machining processes. Manufacturing defects such as size mismatches and improper fastener application can lead to residual stresses and even cracks, both strong stress concentrations.
Damage and Exposure	Damages during service life can lead a part to failure. Damages such as cracks, debonding, and delamination can result from unanticipated resonant vibrations and impacts that exceed the design loads. Reduction in strength can result from exposure to UV lights and chemical corrosion.
Fatigue and Creep	Fatigue or creep can lead a part to failure. For example, unanticipated fatigue can result from repeated mechanical or thermal loading.

Reference:

http://www.samconsult.biz/Science/Failure_Criteria/Failure%20Criteria.htm